Lecture 23 Intro. to the Heat and Wave Equations!

Happy Thanksgiving!

Warmup! Find basis of solns of

$$\underline{y}' = \underbrace{\begin{bmatrix} -1 & 2 \\ -1 & -3 \end{bmatrix}}_{A} \underline{y} \qquad \underline{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Soln

Find e-values/e-vectors of A

$$\chi_A(r) = \det(A - r\,Id)$$
$$= r^2 + 4r + 5$$

Quad formula

$$r_{\pm} = \frac{-4 \pm \sqrt{16-20}}{2} = -2 \pm i$$

$$r_1 = -2+i \quad , \quad r_2 = -2-i$$

Now e-vectors :

$$(A - r_1 \mathrm{Id}) = \begin{bmatrix} 1-i & 2 \\ -1 & -1-i \end{bmatrix}$$

$$\underline{u}_1 = \begin{bmatrix} -2 \\ 1-i \end{bmatrix} \quad \text{e-vector for } r_1 = -2+i$$

$$\underline{u}_2 = \overline{\underline{u}_1} = \begin{bmatrix} -2 \\ 1+i \end{bmatrix} \quad \text{e-vector for } r_2 = \overline{r_1} = -2-i$$

Basis of solns

$$e^{(-2+i)t} \cdot \begin{bmatrix} -2 \\ 1-i \end{bmatrix}, \quad e^{(-2-i)t} \begin{bmatrix} -2 \\ 1+i \end{bmatrix}$$

Let's take one more step and find basis of <u>real</u> fns.

We'll use method of taking real and imag. part of one of solns.

$$e^{(-2+i)t}\begin{bmatrix}-2\\1-i\end{bmatrix} = e^{-2t}e^{it}\begin{bmatrix}-2\\1-i\end{bmatrix}$$

$$= e^{-2t}\left(\cos t + i\sin t\right)\begin{bmatrix}-2\\1-i\end{bmatrix}$$

$$= e^{-2t}\left[\left(\cos t\begin{bmatrix}-2\\1\end{bmatrix} + \sin t\begin{bmatrix}0\\1\end{bmatrix}\right) + \dots\right.$$

$$\ldots + i\left(\sin t \begin{bmatrix} -2 \\ 1 \end{bmatrix} - \cos t \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)\Bigg]$$

__Real part:__ $e^{-2t}\left(\cos t \begin{bmatrix} -2 \\ 1 \end{bmatrix} + \sin t \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$

__Imag part:__ $e^{-2t}\left(\sin t \begin{bmatrix} -2 \\ 1 \end{bmatrix} - \cos t \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$

These two fns form real basis of solns.

Warmup 2 Find basis of solns to

$$\underline{y}' = \underbrace{\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}}_{A} \underline{y} \qquad \underline{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Soln

Find e-values / e-vectors of A

$r = -1$ only e-value.

~~Only~~ Basis of e-vectors $\underline{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

So we've found 1 soln

$$e^{-t} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

We need 2 more solns!

<u>Amazing</u> <u>method to do this</u>:

Try $e^{At} = Id + At + \frac{(At)^2}{2!} + \dots$

Take <u>derivative</u>:

$$\left(e^{At}\right)' = A e^{At} \quad !$$

<u>Conclusion</u>:

Columns of e^{At} form basis of solns

$$\cdots + \frac{1}{2}\begin{pmatrix} 0 & 0 & t^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

higher order terms are all 0.

We find

$$e^{At} = \begin{pmatrix} e^{-t} & & \\ & e^{-t} & \\ & & e^{-t} \end{pmatrix} \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{pmatrix} e^{-t} & te^{-t} & \frac{t^2}{2}e^{-t} \\ 0 & e^{-t} & te^{-t} \\ 0 & 0 & e^{-t} \end{pmatrix}$$

Back to exercise: $A = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$

$$e^{At} = e^{\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} t} e^{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} t}$$

$$= \begin{pmatrix} e^{-t} & & \\ & e^{-t} & \\ & & e^{-t} \end{pmatrix} \cdot e^{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} t}$$

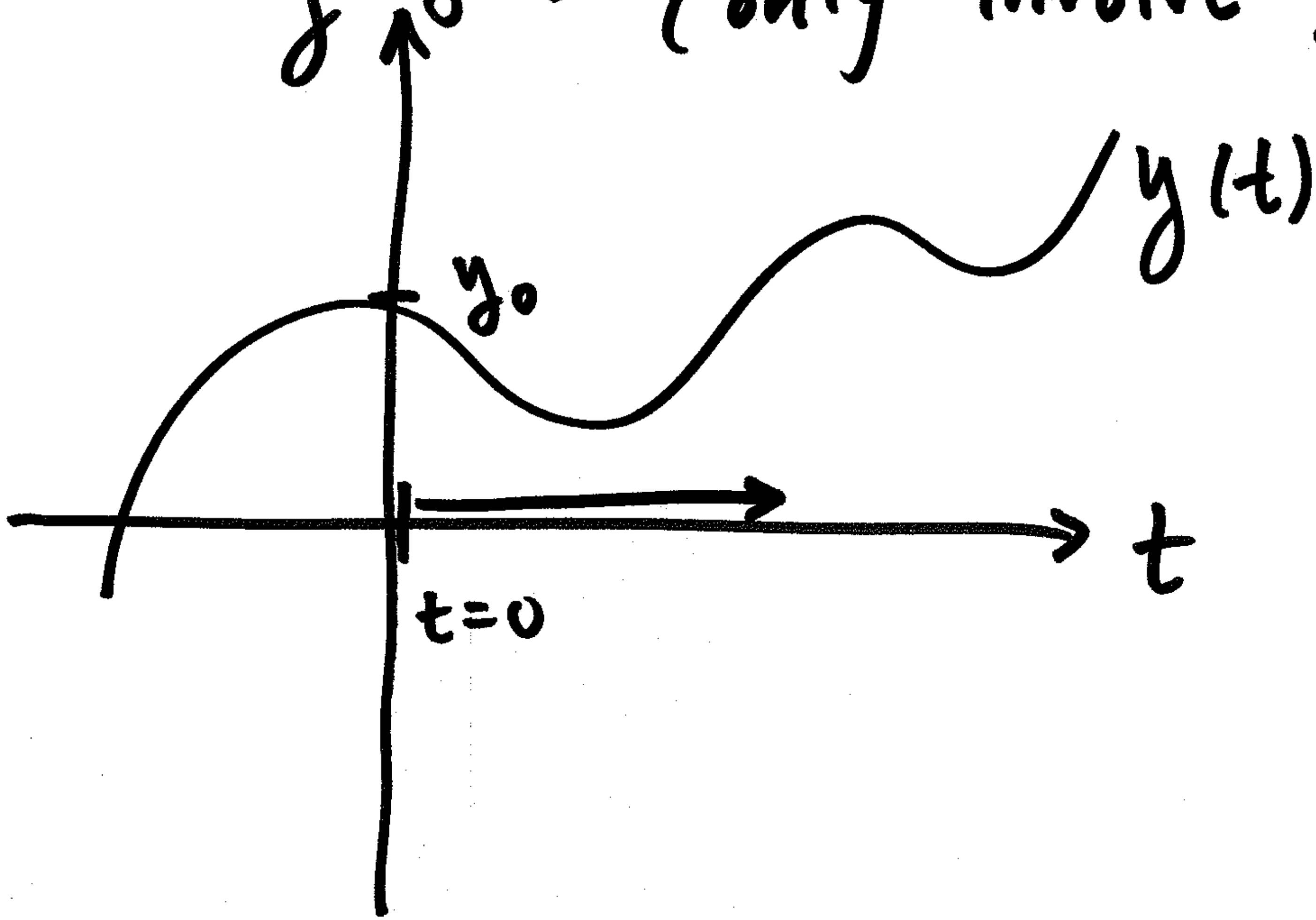
Second term:

$$e^{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \cdot t} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} + \begin{pmatrix} 0 & t & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{pmatrix} + \ldots$$

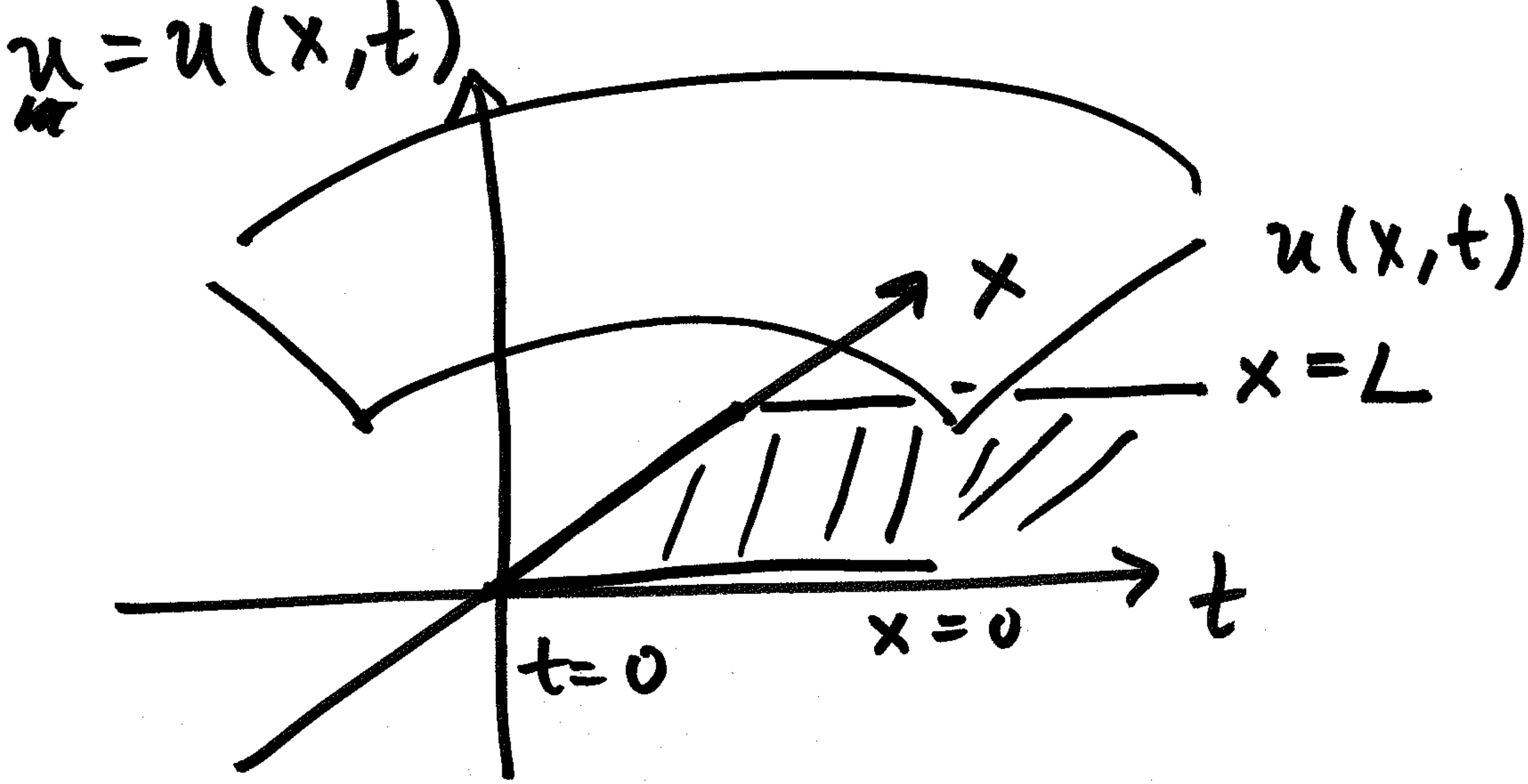
Basis of solns given by cols:

$$\begin{bmatrix} e^{-t} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} te^{-t} \\ e^{-t} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{t^2}{2}e^{-t} \\ te^{-t} \\ e^{-t} \end{bmatrix}$$

We have been studying fns of <u>one</u> var t and ODE (only involve $\frac{d}{dt}$)



Now we will study fns of <u>two</u> (or more) vars t, x and PDE (involve $\frac{\partial}{\partial t}, \frac{\partial}{\partial x}$)



Heat Equation: $u = u(x,t)$

temperature

$$\frac{\partial u}{\partial t} = \beta \frac{\partial^2}{\partial x^2} u$$

$$u(x,0) = f(x)$$

initial value

$$u(0,t) = 0 = u(L,t)$$

boundary values

Wave Equation: $u = u(x,t)$ displacement

$$\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

$$\left.\begin{aligned} u(x,0) &= f(x) \\ \frac{\partial u}{\partial t}(x,0) &= g(x) \end{aligned}\right\}$$

initial values

$$u(0,t) = 0 = u(L,t)$$

boundary values