Lecture 20 Nonhomogeneous Second Order Diff. Eqns

Today: Off. Hours 12:15 - 2 pm
736 Evans

Friday: Quiz through 4.3

Warmup Solve I.V.P. for $y = y(t)$ where

$$y'' - 4y' + 5y = 0$$

$$y(0) = 3 \quad y'(0) = -4$$

Soln: Rewrite eqn

$$\underbrace{\left(\left(\frac{d}{dt}\right)^2 - 4\frac{d}{dt} + 5\right)}_{\text{lin transf}} y = 0$$

vectors

Set $r = \frac{d}{dt}$ to obtain aux. eqn

$$r^2 - 4r + 5 = 0$$

Factor : findroots $r = 2 \pm i$

(using quad formula)

Rewrite eqn

$$\left(\frac{d}{dt} - (2+i)\right)\left(\frac{d}{dt} - (2-i)\right)y = 0$$

(order doesn't matter)

Basis of soln set

$$e^{(2+i)t}, \; e^{(2-i)t}$$

How to express only in real ~~num~~ numbers?

$$e^{(2+i)t} = e^{2t} e^{it}$$

$$= e^{2t}(\cos t + i \sin t)$$

$$e^{(2-i)t} = e^{2t} e^{-it}$$

$$= e^{2t}(\cos(-t) + i \sin(-t))$$

$$= e^{2t}(\cos(t) - i \sin(t))$$

New basis

$$\frac{e^{(2+i)t} + e^{(2-i)t}}{2} = e^{2t}\cos t \quad , \quad \frac{e^{(2+i)t} - e^{(2-i)t}}{2i} = e^{2t}\sin t$$

Gen soln:

$$y = c_1 e^{2t}\cos t + c_2 e^{2t}\sin t$$

Calculate from gen soln

	Want
$y(0) = c_1$	$= 3$
$y'(0) = 2c_1 + c_2$	$= -4$

Solve lin syst

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

$c_1 = 3 \quad c_2 = -10$

Soln to IVP

$$y = 3e^{2t}\cos(t) - 10e^{2t}\sin(t)$$

We now know how to solve second order homogeneous diff eqns.

What can we say about <u>nonhomogeneous</u> second order diff eqn?

$$y'' + by' + cy = f(t)$$

$$\underbrace{\left(\left(\frac{d}{dt}\right)^2 + b\frac{d}{dt} + c\right)}_{\text{lin transf}} \cdot y = f(t)$$

vectors.

If we can find <u>one</u> soln y,
then any soln will be of
the form

$$y + c_1 y_1 + c_2 y_2$$

where y_1, y_2 basis of solns
to homogeneous eqn

<u>Exer</u> Find one soln to

$$y'' + 2y' + y = t^2$$

<u>Soln</u> Oldest scientific method...

<u>Educated</u> Guess!

<u>Try</u>

$$t^2 \longmapsto 2 + 4t + t^2$$

$$t \longmapsto 0 + 2 + t$$

$$1 \longmapsto 0 + 0 + 1$$

<u>Observe</u> t^2 is in span of

$2+4t+t^2,\ 2+t,\ 1$!

<u>Try</u> $y = A_2 t^2 + A_1 t + A_0$

and solve lin syst for A_2, A_1, A_0

$$\begin{array}{c} t^2 \\ t \\ 1 \end{array} \begin{bmatrix} \cancel{2}\,1 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} A_2 \\ A_1 \\ A_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Find $A_2 = 1$, $A_1 = -4$, $A_0 = 6$

$y = 1 \cdot t^2 - 4 \cdot t + 6$ is a soln!

More generally, to solve

$$y'' + by' + cy = t^m$$

Guess $y = A_m t^m + \cdots + A_0$

since t^m will be in span of diff eqn applied to

$$t^m, t^{m-1}, \ldots, 1$$

Then solve lin syst

$$\begin{array}{c} t^m \\ t^{m-1} \\ \vdots \\ \vdots \\ 1 \end{array} \begin{bmatrix} | & & | \\ \underline{v}_m & \cdots & \underline{v}_0 \\ | & & | \end{bmatrix} \begin{pmatrix} A_m \\ A_{m-1} \\ \vdots \\ A_0 \end{pmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

where $\underline{v}_k$ is output of diff eqn applied to t^k

<u>Exer</u> Find a soln to

$$y'' + 3y' + 2y = e^{3t}$$

Basis for solns to homog. eqn

$$e^{-t}, e^{-2t}$$

Guess for nonhomog eqn $y = e^{3t}$

$$e^{3t} \longmapsto 9e^{3t} + 9e^{3t} + 2e^{3t}$$

$$= 20e^{3t}$$

Take $y = \frac{e^{3t}}{20}$

Exer Find a soln to

$$\cdot \; \_\_\_\_ \; \cdot = e^{-2t}$$

Guess for nonhomog eqn $y = e^{-2t}$

But $e^{-2t} \longmapsto 0$ since it solves homog eqn

Next guess $y = te^{-2t}$

$$
\begin{aligned}
te^{-2t} \longmapsto\ & \frac{d}{dt}\left(e^{-2t} + (-2)te^{-2t}\right) \\
& + 3\left(e^{-2t} + (-2)te^{-2t}\right) \\
& + 2te^{-2t} \\
& = -2e^{-2t} - 2e^{-2t} + \cancel{4te^{-2t}} \\
& + 3e^{-2t} - \cancel{6te^{-2t}} \\
& + \cancel{2te^{-2t}} \\
& = -e^{-2t}
\end{aligned}
$$

Take $y = -te^{-2t}$ to solve

$$y'' + 3y' + 2y = e^{-2t}$$

More generally to solve

$$y'' + by' + cy = e^{rt}$$

r not a root of aux eqn	$y = ce^{rt}$
r a simple root of aux eqn	$y = cte^{rt}$
r a repeated root of aux eqn	$y = ct^2e^{rt}$

Exer Find a soln to

$$y'' - 2y' + y = te^t$$

Basis to solns of homog eqn

$$e^t, \; te^t$$

Guess $t^2e^t \longmapsto \frac{d}{dt}(2te^t + t^2e^t)$

$$-2(2te^t + t^2e^t)$$

$$+ t^2e^t$$

$$= 2e^t + 2te^t + 2te^t + \cancel{t^2 e^t}$$

$$- 4te^t - \cancel{2t^2 e^t}$$

$$+ \cancel{t^2 e^t}$$

$$= 2e^t$$

Thaka

Guess $t^3e^t \longmapsto \frac{d}{dt}(3t^2e^t + t^3e^t)$

$$-2(3t^2e^t + t^3e^t)$$

$$+ t^3e^t$$

$$= 6te^t + \cancel{3t^2e^t} + \cancel{3t^2e^t} + \cancel{t^3e^t}$$

$$- \cancel{6t^2e^t} - \cancel{2t^3e^t} + \cancel{t^3e^t}$$

$$= 6te^t$$

Take $y = \frac{t^3e^t}{6}$

__General story__: to solve

$$y'' + by' + cy = t^m e^{rt}$$

__r not a root of aux eqn__

$$y = (A_m t^m + \cdots + A_0) e^{rt}$$

solve lin syst for A_i's

__r simple root__

$$y = t(A_m t^m + \cdots + A_0) e^{rt}$$

solve lin syst for A_i's

__r repeated root__

$$y = t^2(A_m t^m + \cdots + A_0) e^{rt}$$

solve lin syst for A_i's

Final remarks

1) Superposition principle to solve

$$y'' + by' + cy = f + g$$

Take sum of solns to

$$y'' + by' + cy = f$$

$$y'' + by' + cy = g$$

2) There are also nice formulas for solns to eqn

$$y'' + by' + cy = t^m e^{\alpha t} \cos \beta t$$

$$\text{or } t^m e^{\alpha t} \sin \beta t$$