Lecture 16 : Geometry in $\mathbb{R}^n$

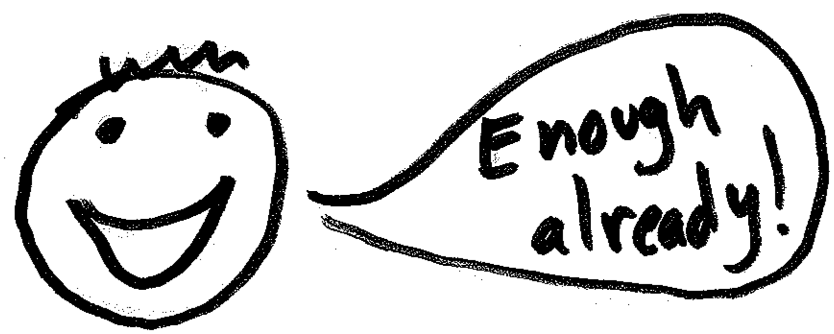
<u>Today</u> : Off Hrs 12-2pm 736 Evans

<u>Friday</u> : Quiz through 6.1

<u>Next Week</u> Review Tuesday 11-2
location TBA

<u>Thursday</u> : Midterm 2 through 6.3

<u>Friday</u> : No Quiz



Recall from last time interpretation of

$$A = PDP^{-1}$$

$$P = P_{\mathcal{B}} = \underset{\mathcal{E} \leftarrow \mathcal{B}}{P} \qquad \underline{x} = \underset{\mathcal{E} \leftarrow \mathcal{B}}{P} [\underline{x}]_{\mathcal{B}}$$

$\mathcal{B} = \{ \underline{b}_1, \dots, \underline{b}_n \}$ basis of e-vectors

So we can think of eqn as saying that D is the matrix of A wrt $\mathcal{B}$-coordinates

Warmup Are $A = \begin{bmatrix} -2 & 3 \\ -1 & 2 \end{bmatrix}$ and $A' = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$

similar?

We are asking: is there an invertible matrix R so that $A = RA'R^{-1}$?

Soln: strategy is to diagonalize. (or at least try to)

$$\det(A - \lambda) = (-2-\lambda)(2-\lambda) + 3 = \lambda^2 - 1$$

$$\det(A' - \lambda) = \lambda^2 - 1$$

$$(A-1) = \begin{bmatrix} -3 & 3 \\ -1 & 1 \end{bmatrix} \qquad \underline{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$(A-(-1)) = \begin{bmatrix} -1 & 3 \\ -1 & 3 \end{bmatrix} \qquad \underline{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\left.\begin{matrix} \underline{v}_1, \underline{v}_2 \end{matrix}\right\} B$$

$$(A'-1) = \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} \qquad \underline{u}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$(A'-(-1)) = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} \qquad \underline{u}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\left.\begin{matrix} \underline{u}_1, \underline{u}_2 \end{matrix}\right\} B'$$

$$D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad P = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} \quad P' = \begin{pmatrix} 0 & -2 \\ 1 & 1 \end{pmatrix}$$

$A = PDP^{-1} \qquad A' = P'D(P')^{-1}$

So $P^{-1}AP = D \qquad (P')^{-1}A'P' = D$

So $P^{-1}AP = (P')^{-1}A'P'$

Finally $A = P(P')^{-1}A'P'P^{-1}$

Set $R = P(P')^{-1}$ Then $R^{-1} = P'P^{-1}$

So we've shown

$$A = RA'R^{-1}$$

More generally $T : V \to W$ lin transf can be represented as a matrix if we choose bases

$\mathcal{B} = \{\underline{b}_1, \dots, \underline{b}_n\}$ of V $\qquad$ $\mathcal{C} = \{\underline{c}_1, \dots, \underline{c}_m\}$ of W

$$[T] = \begin{bmatrix} | & & | \\ [T(\underline{b}_1)]_{\mathcal{C}} & \cdots & [T(\underline{b}_n)]_{\mathcal{C}} \\ | & & | \end{bmatrix}$$

$m \times n$ matrix

Satisfies

$$[T][\underline{x}]_B = [T\underline{x}]_C$$

Pictre

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "{[-]_B}"'] & W \arrow[d, "{[-]_C}"] \\ \mathbb{R}^n \arrow[r, "{[T]}"] & \mathbb{R}^m \end{tikzcd}$$

<u>Exer</u> $T : P_2 \longrightarrow P_2 \quad T(p(x)) = p'(x) + p(0) \cdot x$

1) Calculate $[T]$ wrt standard basis

$$B = C = \{1, x, x^2\}$$

$$T(1) = x \quad T(x) = 1 \quad T(x^2) = 2x$$

$$[T] = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

2) Choose basis $B = \{1, x, x^2\}$

but choose C wisely so that

$$[T] = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C = \{x, 1, x^2\}$$

3) Choose both B and C wisely so that

$$[T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{lccc} B = \{ & 1, & x, & 2-x^2 \} \\ & \downarrow T & \downarrow T & \downarrow T \\ & & & 0 \\ C = \{ & x, & 1, & x^2 \} \end{array}$$

<u>Geometry in $\mathbb{R}^n$</u> Up to now, never discussed <u>length</u>, <u>angle</u> (though we did discuss <u>volume</u>)

<u>Def</u> $\underline{u}, \underline{v}$ vectors in $\mathbb{R}^n$, <u>inner product</u> or <u>dot product</u> is given by

$$\underline{u} \cdot \underline{v} = u_1 v_1 + \cdots + u_n v_n$$

$$\underline{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \qquad \underline{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

Note $\underline{u} \cdot \underline{v} = \underline{u}^T \underline{v}$

$$= \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

$$= \begin{bmatrix} u_1 v_1 + \cdots + u_n v_n \end{bmatrix}$$

Def **Length** of a vector $\underline{u}$ in $\mathbb{R}^n$ is given by

$$\|\underline{u}\| = \sqrt{\underline{u}\cdot\underline{u}} = \sqrt{u_1^2+\cdots+u_n^2}$$

$$\underline{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

Note: $\|\underline{u}\|^2 = \underline{u}\cdot\underline{u}$

Key properties

1) $\underline{u} \cdot \underline{v} = \underline{v} \cdot \underline{u}$

2) $(\underline{u} + \underline{v}) \cdot \underline{w} = \underline{u} \cdot \underline{w} + \underline{v} \cdot \underline{w}$

3) $(c\underline{u}) \cdot \underline{v} = c(\underline{u} \cdot \underline{v}) = \underline{u} \cdot (c\underline{v})$

4) $\underline{u} \cdot \underline{u} \geqslant 0$

$\quad (= 0 \iff \underline{u} = \underline{0})$

Def. Unit vector $\underline{u}$ vector in $\mathbb{R}^n$

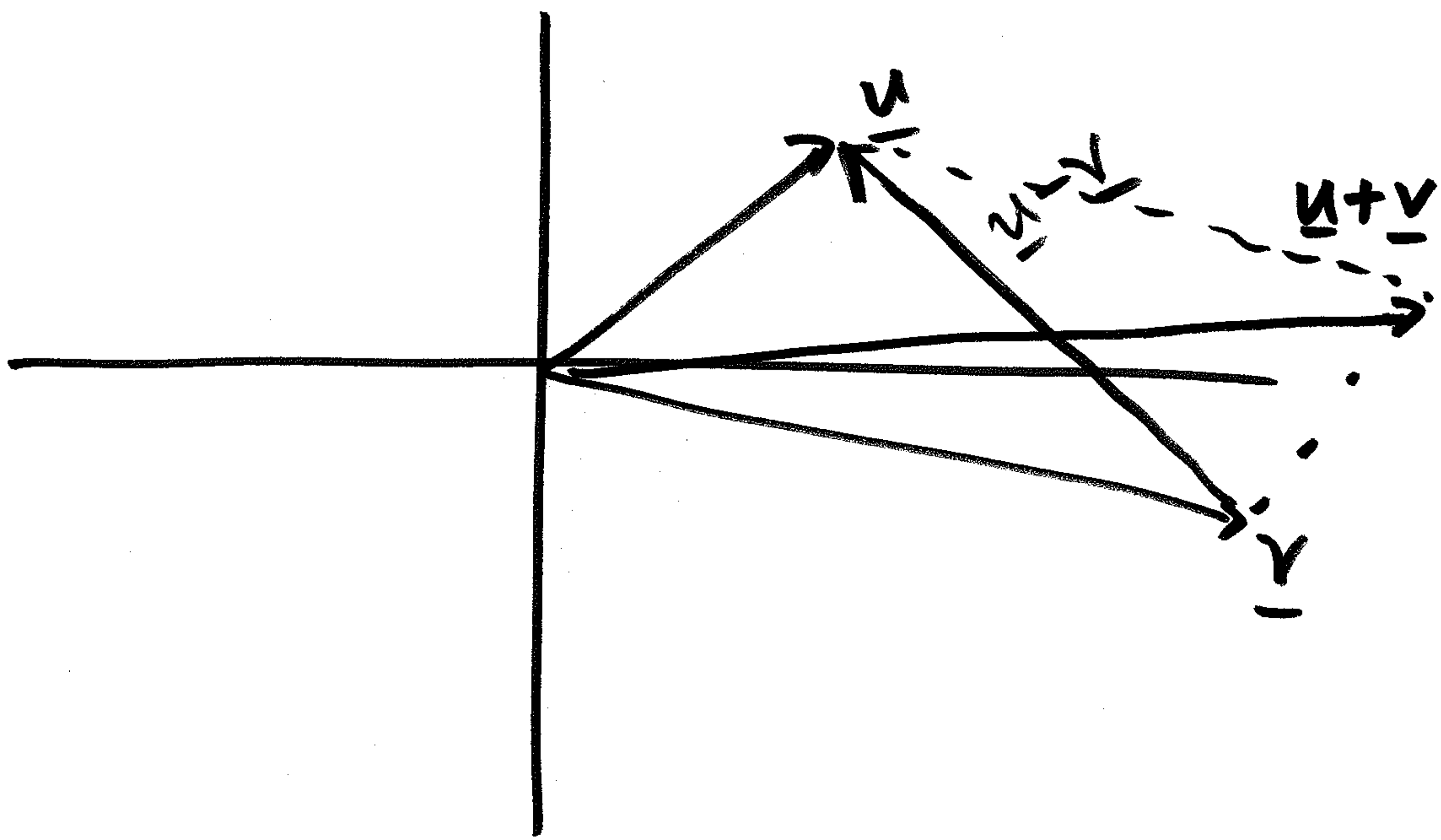
with $\|\underline{u}\| = 1$

(equivalently $\underline{u} \cdot \underline{u} = 1$)

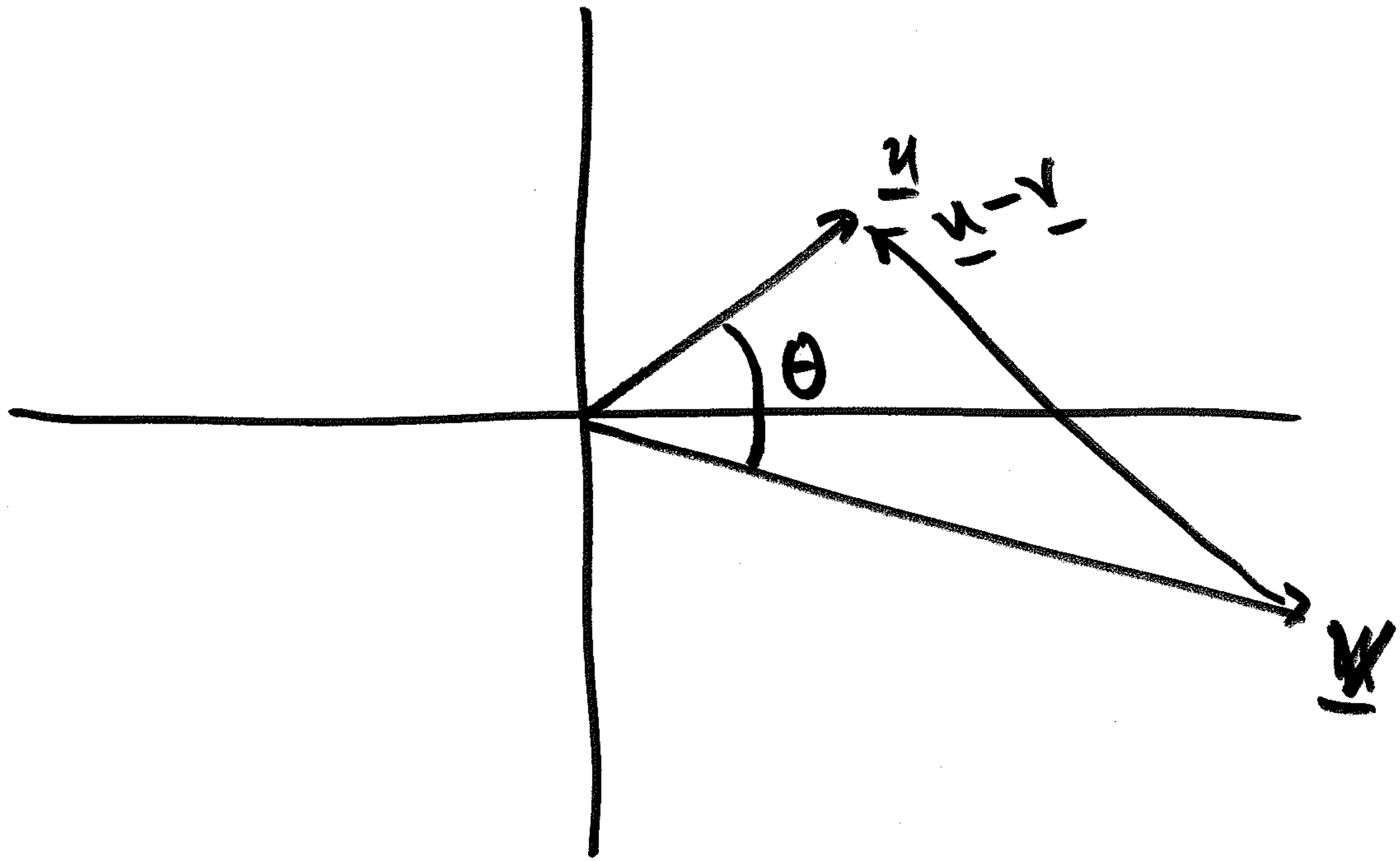
Given $\underline{u} \neq \underline{0}$ there is unique unit vector in same direction

$$\underline{u} \rightsquigarrow \frac{\underline{u}}{\|\underline{u}\|}$$ normalization

Def distance $\text{dist}(\underline{u}, \underline{v}) = \|\underline{u} - \underline{v}\|$



What does $\underline{u} \cdot \underline{v}$ mean geometrically?



$$\|\underline{u}-\underline{v}\|^2 = (\underline{u}-\underline{v})\cdot(\underline{u}-\underline{v}) = \underline{u}\cdot\underline{u} + \underline{v}\cdot\underline{v} - 2\,\underline{u}\cdot\underline{v}$$

Law of cosines tells you

$$\underline{u} \cdot \underline{v} = \|\underline{u}\| \cdot \|\underline{v}\| \cos \theta$$

<u>Def.</u> $\underline{u}, \underline{v}$ are <u>orthogonal</u>

if $\underline{u} \cdot \underline{v} = 0$.

<u>Pythagorean Thm</u> If $\underline{u}, \underline{v}$ are orthog.

then $\|\underline{u} + \underline{v}\|^2 = \|\underline{u}\|^2 + \|\underline{v}\|^2$

__Def.__ W in $\mathbb{R}^n$ is a subspace, we define $W^\perp$ in $\mathbb{R}^n$ to consist of all $\underline{v}$ in $\mathbb{R}^n$ such that $\underline{v}$ is orthog. to all $\underline{w}$ in W.

__Exer.__ 1) $W^\perp$ is a subspace

2) If $W = \text{Span}\{\underline{v}_1, \ldots, \underline{v}_k\}$ then $\underline{v}$ is in $W^\perp \iff \underline{v}$ orthog. to $\underline{v}_i$ all i

__Example__ A $m \times n$ matrix

Then recall $\text{Col } A$ = Span of cols

$\text{Row } A$ = Span of rows

Then $(\text{Row } A)^{\perp} = \text{Nul } A$

$(\text{Col } A)^{\perp} = \text{Nul } (A^T)$

__Def__. 1) $\underline{v}_1, \dots, \underline{v}_k$ in $\mathbb{R}$ is __orthog. set__

if $\underline{v}_i$ orthog $\underline{v}_j$ all $i \neq j$

(pairwise orthog.)

2) $\underline{v}_1, \dots, \underline{v}_k$ in $\mathbb{R}^n$ is __orthonormal set__

if orthog. and

$\|v_i\| = 1$ all i

Thm If $\underline{v}_1, \ldots, \underline{v}_k$ in $\mathbb{R}^n$ is an orthog set of nonzero vectors, then it is lin indep.

Def. $\underline{v}_1, \ldots, \underline{v}_n$ in $\mathbb{R}^n$ is an

- orthog. basis if orthog and basis
- orthonormal basis if orthonormal and basis

Note: if n vectors are orthonormal then they are a basis!