

Lecture 15 Diagonalization as Change of Basis

This week: Thurs Off Hrs 12-2 pm
736 Evans

Fri Quiz through 6.1

Next week: Tues Review 11-2
TBA

Thur Midterm 2
through 6.3
Fri No quiz 😊 phew!

Warmup!

Decide if

$$A = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{pmatrix}$$

is 1) diagonalizable?

2) invertible?

Recall: A is diagonalizable if there are matrices P, D so that P is invertible and D is diagonal

$$A = P D P^{-1}$$

(Equivalently $AP = PD$)

Step 1 Find e-values.

Find solns of $\chi_A(\lambda) = \det(A - \lambda) = 0$.

$$A - \lambda = \begin{vmatrix} -\lambda & -1 & -1 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & -\lambda \end{vmatrix}$$

E-values

$$\lambda = 0, 1$$

mult \uparrow mult \uparrow
1 2

$$\begin{aligned} \chi_A(\lambda) &= -\lambda^3 + 2\lambda^2 - \lambda \\ &= -\lambda(\lambda^2 - 2\lambda + 1) \\ &= -\lambda(\lambda - 1)^2 \end{aligned}$$

Step 2 Find e-vectors.

$$\underline{\lambda=0} \quad A-0 = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Basis for $\text{Null}(A-0) \quad \underline{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$$\underline{\lambda = 1} \quad A - I = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Basis for $\text{Null}(A - I)$ $\underline{v_2} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ $\underline{v_3} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Step 3 Concatenate into single list

of e-vectors

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$v_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$



We found basis!

(enough e-vectors)
 $n=3$

$$D = \begin{bmatrix} 0 & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Remark: We saw that if $n \times n$ matrix A has n distinct e-values, then

A is diagonalizable.

But it is not necessary for A to have n distinct e-values for it to be ~~invertible~~ diagonalizable!

Is A invertible? No!

Reason 1 0 is e-value of A .

Reason 2 Suppose A were invertible

We've seen $A = PDP^{-1}$

So $D = P^{-1}AP$

Then D would be invertible

$D^{-1} = P^{-1}A^{-1}P$ But $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
↳

Warmup 2 Show $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ is

1) not diagonalizable.

2) invertible. $A^{-1} = \frac{1}{4} \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}$

Step 1 Find e-values

$\lambda = 2$. — A is Δ -ar.
so diag entries
are e-values

mult = 2 — $\chi_A(\lambda) = (2 - \lambda)^2$

Step 2 Find e-vectors

$$A - 2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \dim \text{Nul}(A - 2) = 1$$

Basis $\underline{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Not diagonalizable: not enough e-vectors.

Let's see what it would mean if

$$A = PDP^{-1}, \quad D \text{ diagonal}$$

Exer Show cols of P are e -vectors

for A , using eqn
 $A = PDP^{-1}$ and that
 D is diagonal

So cols of P would form a
basis of e -vectors.

Challenge What's the closest we can

get to diagonalizing $A = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}$?

$$\begin{aligned}\chi_A(\lambda) &= \det(A - \lambda) \\ &= (1 - \lambda)(3 - \lambda) + 2 \\ &= \lambda^2 - 4\lambda + 5\end{aligned}$$

$$\lambda = \frac{4 \pm \sqrt{16 - 20}}{2} = 2 \pm i$$

$$\lambda_+ = 2 + i, \lambda_- = 2 - i \quad \begin{array}{l} \text{Re} = 2 \\ \text{Im} = \pm 1 \end{array}$$

If we insist on only real matrices,

A is not diagonalizable.

But if we allow complex numbers

we should take $D = \begin{pmatrix} 2+i & 0 \\ 0 & 2-i \end{pmatrix}$

(Always find $\lambda_- = \overline{\lambda_+}$)

Find complex e-vectors \bar{v}_+ , \bar{v}_-

$$A - \lambda_+ = \begin{bmatrix} -1-i & -2 \\ 1 & 1-i \end{bmatrix}$$

nontriv.

We know there is null space

$$\bar{v}_+ = \begin{bmatrix} -2 \\ 1+i \end{bmatrix}$$

Check:

$$(A - \lambda_+) \bar{v}_+ = \bar{0}.$$

Similarly ... find \underline{v}_-

$$A - \lambda_- = \begin{bmatrix} -1+i & -2 \\ 1 & 1+i \end{bmatrix}$$

$$\underline{v}_- = \begin{bmatrix} -2 \\ 1-i \end{bmatrix}$$

Check:

$$(A - \lambda_-) \underline{v}_- = \underline{0}$$

(Always take $\underline{v}_- = \overline{\underline{v}_+}$)

What if we're Neanderthals and don't accept complex numbers?

Best alternative: Find P, C

invertible \rightarrow

rotation + scaling

so that

$$A = PCP^{-1}$$

$$C = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

$$\begin{aligned} r &= \sqrt{a^2 + b^2} \\ a &= r \cos \theta \\ b &= r \sin \theta \end{aligned}$$

$$= \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

Back to challenge problem:

$$\lambda_+ = 2 + i \quad \bar{\lambda}_+ = \begin{bmatrix} -2 \\ 1+i \end{bmatrix}$$

\uparrow \uparrow
a b

Take

$$C = \begin{bmatrix} 2 & +1 \\ -1 & 2 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1 \\ \operatorname{Re}(\bar{\lambda}_+) & \operatorname{Im}(\bar{\lambda}_+) \end{bmatrix}$$
$$= \begin{bmatrix} -2 & 0 \\ 1 & 1 \end{bmatrix}$$

Note $C = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$= \begin{bmatrix} \sqrt{5} & 0 \\ 0 & \sqrt{5} \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & +\frac{1}{\sqrt{5}} \\ -\frac{4}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$$



Scale



rotation

Conceptual meaning of diagonalization

What does $A = PDP^{-1}$ mean?

Def. $n \times n$ matrices A, A' are similar if there is an invertible $n \times n$ matrix P so that

$$A = PA'P^{-1}$$

You should think of similar matrices as "the same transf" from different perspectives.

Let's think of the cols of P

as a basis $B = \{v_1, \dots, v_n\}$

$$\text{Then } P = P_B = \sum_{\xi \in B} \xi$$

$$\underline{x} = \sum_{\xi \in B} \xi [\underline{x}]_B$$

$$\text{And } P^{-1} = \sum_{\xi \in B} \xi$$

$$[\underline{x}]_B = \sum_{\xi \in B} \xi^{-1} \underline{x}$$