Welcome to Lecture 14!

Where we learn why we

<u>love</u> eigenvalues & eigenvectors

<u>Today</u>: Off. Hrs 12-2pm 736 Evans

<u>Friday</u> Quiz thru 5.2

<u>Warmup</u> Find e-values and bases for e-spaces of matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -2 & 0 \\ 1 & 4 & 2 \end{bmatrix}$$

Find λ so that $A-\lambda$ <u>not</u> invt-ble

Calc. $\det(A-\lambda)$

$$A - \lambda I = \begin{bmatrix} -\lambda & 1 & 0 \\ -1 & -2-\lambda & 0 \\ 1 & 4 & 2-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (2-\lambda)(\lambda^2 + 2\lambda + 1)$$
$$= (2-\lambda)(\lambda+1)^2$$

E-<u>values</u> $\lambda = 2, -1$

Now find e-space for $\underline{\lambda = 2}$.

Null$(A - 2I)$

$$A - 2I = \begin{bmatrix} -2 & 1 & 0 \\ -1 & -4 & 0 \\ 1 & 4 & 0 \end{bmatrix}$$

$$\xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Basis for $\lambda = 2$ e-space

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Find e-space for $\underline{\lambda = -1}$.

$$A - (-1)I = \begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 1 & 4 & 3 \end{bmatrix}$$

$$\underset{\text{RREF}}{\rightsquigarrow} \begin{array}{c} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \text{pivot} & \text{pivot} & \text{free} \end{array} \end{array}$$

Basis $\lambda = -1$ e-space

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Def. A $n \times n$ matrix

1) Characteristic polynomial of A is

$$\chi_A(\lambda) = \det(A - \lambda I)$$

2) Characteristic equation of A is

$$\chi_A(\lambda) = 0$$

__Observe__: __roots__ of $\chi_A(\lambda)$, in other words __solns__ of $\chi_A(\lambda)=0$, are the __e-values__ of A

__Fact__: __multiplicity__ of a root λ is $\geqslant$ dim of λ e-space.

Thm Suppose A $n \times n$ matrix

$\lambda_1, \dots, \lambda_k$ k distinct e-values

Then the corr. e-vectors $\underline{v}_1, \dots, \underline{v}_k$

are lin indep.

Proof. Suppose not. Then there is

an $\ell \le k$ so that

$$\underline{v}_\ell = a_1 \underline{v}_1 + \dots + a_{\ell-1} \underline{v}_{\ell-1}$$

and $\underline{v}_1, \dots, \underline{v}_{\ell-1}$ are lin indep.

Apply A:

$$\lambda_\ell \underline{v}_\ell = a_1 \lambda_1 \underline{v}_1 + \cdots + a_{\ell-1} \lambda_{\ell-1} \underline{v}_{\ell-1}$$

So:

$$\underline{0} = a_1(\lambda_1 - \lambda_\ell)\underline{v}_1 + \cdots + a_{\ell-1}(\lambda_{\ell-1} - \lambda_\ell)\underline{v}_{\ell-1}$$

Since $\underline{v}_1, \ldots, \underline{v}_{\ell-1}$ are lin indep,

we have

$$a_1(\lambda_1 - \lambda_\ell) = \cdots = a_{\ell-1}(\lambda_{\ell-1} - \lambda_\ell) = 0$$

If $a_1 = \cdots = a_{\ell-1} = 0$, then $\underline{v}_\ell = \underline{0}$

but $\lightning$ since $\underline{v}_\ell$ is an e-vector.

So some $a_i \neq 0$. So $\lambda_i - \lambda_\ell = 0$.

but $\lightning$ since λ_i's are distinct.

Consequences:

1) There can be at most n lin indep vectors

So there can be at most n e-values.

(Also: can see this from the fact that a deg n poly has at most n roots)

2) If we have n distinct e-values, then corr. e-vectors form a basis!

Let's explore what good comes of a basis $\mathcal{B} = \{\underline{v}_1, \ldots, \underline{v}_n\}$ of e-vectors.

<u>Recall</u>: change of coord matrix

$$P_{\mathcal{B}} = \underset{\mathcal{E} \leftarrow \mathcal{B}}{P} = \begin{bmatrix} | & & | \\ \underline{v}_1 & \cdots & \underline{v}_n \\ | & & | \end{bmatrix}$$

<u>Then</u> $D = P_{\mathcal{B}}^{-1} A P_{\mathcal{B}}$ is diagonal!

<u>Check this:</u> D is diagonal means

$$D = \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix}$$

So in other words $D\underline{e}_i = d_i \underline{e}_i$

$$D\underline{e}_i = P_B^{-1} A P_B \underline{e}_i = P_B^{-1} A \underline{v}_i$$
$$= P_B^{-1} \lambda_i \underline{v}_i = \lambda_i P_B^{-1} \underline{v}_i$$
$$= \lambda_i \underline{e}_i$$

We're shown

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

with e-values on diag!

<u>Exer</u> Calculate A^{2014} where

$$A = \begin{bmatrix} -3 & -2 \\ 4 & 3 \end{bmatrix}$$

Note: if we can find basis $\mathcal{B} = \{\underline{v}_1, \underline{v}_2\}$ of e-vectors then $D = P_{\mathcal{B}}^{-1} A P_{\mathcal{B}}$ is diagonal

So $A = P_{\mathcal{B}} D P_{\mathcal{B}}^{-1}$

Hence $A^{2014} = (P_{\mathcal{B}} D P_{\mathcal{B}}^{-1})^{2014} = P_{\mathcal{B}} D^{2014} P_{\mathcal{B}}^{-1}$

So we want to find $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

and corr e-vectors $\underline{v}_1, \underline{v}_2$.

$$\det(A - \lambda) = \det\left(\begin{pmatrix} -3-\lambda & -2 \\ 4 & 3-\lambda \end{pmatrix}\right)$$

$$= -9 + \lambda^2 + 8 = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1)$$

<u>E-values</u> $\lambda = 1, -1$

$$D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

In fact we don't need $\underline{v}_1, \underline{v}_2$ explicitly!

$$A^{2014} = P_B D^{2014} P_B^{-1} = P_B I P_B^{-1} = I$$

Done!

General <u>Algorithm</u> for diagonalizing an $n \times n$ matrix A

1) <u>Find e-values</u> — solns of char. eqn. $\det(A - \lambda I) = 0$.

2) <u>Find bases for e-spaces</u> — bases of $\mathrm{Null}(A - \lambda I)$

3) <u>Collect all the bases into a single list</u>

$\underline{v}_1, \ldots, \underline{v}_k$

<u>Possibilities:</u>

1) <u>$k=n$</u> Good situation!

$\underline{v}_1, \dots, \underline{v}_n$ is a basis of e-vectors

$$D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \qquad P = \begin{bmatrix} | & & | \\ \underline{v}_1 & \cdots & \underline{v}_n \\ | & & | \end{bmatrix}$$

2) $\underline{k < n}$ Bad situation

$\underline{v}_1, \ldots, \underline{v}_k$ does not span

not a basis

We say A is <u>not</u> diagonalizable

Examples $(n=2)$

1) 2 distinct e-values λ_1, λ_2

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad P = \begin{bmatrix} | & | \\ \underline{v}_1 & \underline{v}_2 \\ | & | \end{bmatrix}$$

2) 1 e-value only but 2 dim e-space

$$D = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \quad P = \begin{bmatrix} | & | \\ \underline{v}_1 & \underline{v}_2 \\ | & | \end{bmatrix}$$

1) & 2) Diagonalizable!

3) 1 e-value but only 1 dim e-space

<u>ex</u> $A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$

4) no real e-values!

ex $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

3) & 4) Not Diagonalizable!