Name (Last, First): ____________________

Student ID: ____________________

Circle your GSI and section:

| | | |
|---|---|---|
| Scerbo | 8am | 200 Wheeler |
| Scerbo | 9am | 3109 Etcheverry |
| McIvor | 12pm | 3107 Etcheverry |
| McIvor | 11am | 3102 Etcheverry |
| Mannisto | 12pm | 3 Evans |
| Wayman | 1pm | 179 Stanley |
| Wayman | 2pm | 81 Evans |
| Forman | 2pm | 3109 Etcheverry |
| Forman | 4pm | 3105 Etcheverry |
| Melvin | 5pm | 24 Wheeler |
| Melvin | 4pm | 151 Barrows |
| Mannisto | 11am | 3113 Etcheverry |
| McIvor | 2pm | 179 Stanley |

If none of the above, please explain: ____________________

This is a closed book exam, no notes allowed. It consists of 6 problems, each worth 10 points, of which you must complete 5. **Choose one problem not to be graded by crossing it out in the box below.** You must justify every one of your answers unless otherwise directed.

| Problem | Maximum Score | Your Score |
|---|---|---|
| 1 | 10 | |
| 2 | 10 | |
| 3 | 10 | |
| 4 | 10 | |
| 5 | 10 | |
| 6 | 10 | |
| Total Possible | 50 | |

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1. a. State what it means for a list of vectors $v_1, \ldots, v_n$ in a vector space $V$ over a field $F$ to be linearly independent.

For any $c_1, \ldots, c_n \in F$, if we have $c_1 v_1 + \cdots c_n v_n = 0$, then we have $c_1 = \cdots = c_n = 0$.

b. Consider the real vector space $P_{\leq 2}(\mathbb{R})$ of polynomials of degree less than or equal to two with real coefficients.

For what vectors $p(x) \in P_{\leq 2}(\mathbb{R})$ is the list $p(x), p'(x), p''(x)$ linearly independent? (Here $p'(x)$ denotes the derivative of $p(x)$, and $p''(x)$ denotes the second derivative of $p(x)$.)

We must have $p(x) = a_0 + a_1 x + a_2 x^2$, with $a_2 \neq 0$.

To see this, first calculate $p'(x) = a_1 + 2a_2 x$ and $p''(x) = 2a_2$.

On the one hand, if $a_2 \neq 0$, and $c_1 p(x) + c_2 p'(x) + c_3 p''(x) = 0$, then $c_1 = 0$ since else there will be an $x^2$ term in the sum coming from $p(x)$. But we then must also have $c_2 = 0$ since else there will be an $x$ term coming from $p'(x)$. And finally we then must also have $c_3 = 0$ since else there will be a constant term coming from $p(x)$.

On the other hand, if $a_2 = 0$, then $p''(x) = 0$ and so we may take $c_1 = c_2 = 0$ and $c_3 = 1$.

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2. Suppose $T : V \to W$ is an injective linear transformation of finite-dimensional vector spaces. Show there exists a linear transformation $S : W \to V$ such that the composition $ST : V \to V$ is the identity transformation.

Choose a basis $v_1, \ldots, v_n$ of $V$.

Since $T$ is injective, $Tv_1, \ldots, Tv_n$ is linearly independent. To see this, suppose $a_1Tv_1 + \cdots + a_nTv_n = 0$. Since $T$ is linear, we then have $T(a_1v_1 + \cdots + a_nv_n) = 0$. Since $T$ is injective, this implies $a_1v_1 + \cdots + a_nv_n = 0$. Since $v_1, \ldots, v_n$ is linearly independent, this implies $a_1 = \cdots = a_n = 0$.

Since $Tv_1, \ldots, Tv_n$ is linearly independent, we may extend it to a basis

$$Tv_1, \ldots, Tv_n, w_1, \ldots, w_k$$

Then we can define $S : W \to V$ by setting $S(Tv_i) = v_i$ for $i = 1, \ldots, n$, and $S(w_j) = 0$, for $j = 1, \ldots, k$. We have $ST(v_i) = v_i$ for $i = 1, \ldots, n$, and hence $ST$ is the identity transformation.

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3. Let $P(\mathbb{C})$ be the complex vector space of polynomials with complex coefficients.

Consider the map $T : P(\mathbb{C}) \to \mathbb{C}$ that takes a polynomial $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ to the sum of the complex conjugates of its coefficients

$$T(p(z)) = \bar{a}_0 + \bar{a}_1 + \cdots + \bar{a}_n$$

Is $T$ linear? Be sure to justify your answer.

No, $T$ is not linear since it does not preserve scaling:

$$T(ap(z)) = \bar{a}T(p(z))$$

If we take $a = i$ for example, then $\bar{i} = -i$.

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4. Let $V$ be a two-dimensional vector space. Suppose $T : V \to V$ is a linear transformation that is not a scalar multiple of the identity. Prove that there exists a vector $v \in V$ such that the pair of vectors $v, Tv$ form a basis of $V$.

We have proved that if every nonzero vector of $V$ is an eigenvector of $T$ then $T$ must be a scalar multiple of the identity. Thus there must exist $v \neq 0 \in V$ such that $v$ is not an eigenvector. In other words, $Tv$ is not a scalar multiple of $v$. Hence $v, Tv$ must be linearly independent. Since $V$ is two-dimensional, the fact that $v, Tv$ are linearly independent implies thay also span.

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5. Decide if the following assertion is always true or sometimes false. If always true, provide a proof; if sometimes false, provide a counterexample and justify why it is a counterexample.

Assertion: Let $V$ be a finite dimensional complex vector space, $T : V \to V$ a linear transformation, and $U \subset V$ a $T$-invariant subspace. Then there exists a $T$-invariant subspace $W \subset V$ such that $V = U \oplus W$.

It is false. For example, take $V = \mathbb{C}^2$, $U = \text{span}((1,0))$, and

$$T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

If $W$ is a $T$-invariant subspace, then $W$ must be $\{0\}, U$, or $V$. To see this, suppose $W$ is not $\{0\}$ or $V$, so that $W = \text{span}((a,b))$ for some $a, b \in \mathbb{C}$. But $T((a,b)) = (b,0)$, so we must have $b = 0$ and then $a = 0$, a contradiction.

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6. Let $V$ be a vector space of dimension 3, and let $W$ be a vector space of dimension 5. Recall that $L(V, W)$ denotes the vector space of linear transformations from $V$ to $W$.

Show that there cannot exist a linear transformation

$$T : L(V, W) \to L(V, W)$$

such that $\dim \operatorname{null}(T) = \dim \operatorname{range}(T)$.

We have $\dim L(V, W) = (\dim V)(\dim W) = 3 \cdot 5 = 15$.
We also have $\dim L(V, W) = \dim \operatorname{null}(T) + \dim \operatorname{range}(T)$.
Since 15 is odd, we cannot have $\dim \operatorname{null}(T) = \dim \operatorname{range}(T)$.