Name (Last, First): __________________________________________________________

Student ID: ________________________________________________________________

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<td>McIvor</td>
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If none of the above, please explain: _________________________________________

This exam consists of 10 problems, each worth 10 points, of which you must complete 8. **Choose two problems not to be graded by crossing them out in the box below.** You must justify every one of your answers unless otherwise directed.

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1. Let $V$ be a nonzero finite-dimensional real vector space. Suppose $T : V \to V$ is a linear transformation.

Decide if the following assertions are ALWAYS TRUE or SOMETIMES FALSE. You need not justify your answer.

i. There exists an eigenvalue of $T$.  
   \text{F} 

ii. There exists a basis of $V$ such that $T$ is upper-triangular.  
   \text{F} 

iii. $\dim V = \dim \text{null}(T) + \dim \text{range}(T)$  
    \text{T} 

iv. If $v$ and $w$ are colinear, then $Tv$ and $Tw$ are colinear.  
   \text{T} 

v. If $v$ and $w$ are linearly independent, then $Tv$ and $Tw$ are linearly independent.  
   \text{F} 

vi. If $T$ is invertible and $\lambda$ is an eigenvalue of $T$, then $\lambda^{-1}$ is an eigenvalue of $T^{-1}$.  
    \text{T} 

vii. If $T$ is invertible and $v$ is an eigenvector of $T$, then $v$ is an eigenvector of $T^{-1}$.  
     \text{T} 

viii. If $T^2 = 1$, then $T$ has an eigenvalue.  
      \text{T} 

ix. If $T^3 = T^2$, then $T$ has an eigenvalue.  
    \text{T} 

x. If $T^3 = T^2$, then $\text{null}(T) \neq \{0\}$.  
   \text{F}
2. Let \( V \) be an inner product space and \( v_1, \ldots, v_n \) a list of vectors in \( V \).

(a) State what it means for \( v_1, \ldots, v_n \) to be linearly independent. State what it means for \( v_1, \ldots, v_n \) to be orthonormal.

\( v_1, \ldots, v_n \) is linearly independent means whenever \( a_1v_1 + \cdots + a_nv_n = 0 \) for scalars \( a_1, \ldots, a_n \), we have that \( a_1 = \cdots = a_n = 0 \).

\( v_1, \ldots, v_n \) orthonormal means \( \langle v_i, v_j \rangle \) is equal to 0 if \( i \neq j \) and is equal to 1 when \( i = j \).

(b) Prove that if \( v_1, \ldots, v_n \) is orthonormal, then \( v_1, \ldots, v_n \) is linearly independent.

Suppose \( a_1v_1 + \cdots + a_nv_n = 0 \). Then for all \( i = 1, \ldots, n \), we have \( 0 = \langle a_1v_1 + \cdots + a_nv_n, v_i \rangle = a_1\langle v_1, v_i \rangle + \cdots + a_n\langle v_n, v_i \rangle = a_i\langle v_i, v_i \rangle = a_i \). Thus we have that \( a_1 = \cdots = a_n = 0 \).
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3. Let $A \in M_{n \times n}(\mathbb{C})$ be a complex matrix. Consider the subspace $W \subset M_{n \times n}(\mathbb{C})$ given by

$$W = \text{span}\{I, A, A^2, A^3, \ldots, A^k, \ldots\}$$

Prove that

$$\dim W \leq n.$$ 

By the Cayley-Hamilton Theorem, we have $\chi_A(A) = 0$ where $\chi_A(z)$ is the characteristic polynomial. Recall that $\chi_A(z)$ is monic of degree $n$, and thus $A^n$ is in the span of $I, A, \ldots, A^{n-1}$. For any $k \geq 1$, we similarly have that $A^{n+k}$ is in the span of $A^k, A^{k+1}, \ldots, A^{n+k}$. Thus by induction, we have that $A^{n+k}$ is in the span of $I, A, \ldots, A^{n-1}$. 
4. Consider $\mathbb{C}^3$ with the standard Euclidean inner product. Determine whether each of the following operators $T : \mathbb{C}^3 \rightarrow \mathbb{C}^3$ is self-adjoint, normal, or neither. You need not justify your answer.

a. $T$ has eigenvectors $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ with respective eigenvalues $0, 1 + i, 1 - i$.
   Normal but not self-adjoint.

b. $T$ has eigenvectors $(1, i, 0), (1, -i, 0), (0, 0, 1)$ with respective eigenvalues $1, -1, 0$.
   Self-adjoint.

c. $T$ has eigenvectors $(1, 0, 0), (0, i, -i), (1, 1, 1)$ with respective eigenvalues $1, -1, 1$.
   Self-adjoint.

d. $\dim \text{null}(T^2) = 3$, $\dim \text{range}(T) = 1$.
   Neither.

e. $\dim \text{null}(T - i) = 2$, $\dim \text{null}(T) = 1$ with $\text{null}(T - i) \perp \text{null}(T)$.
   Normal but not self-adjoint.
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5. Find a basis for $\mathbb{C}^3$ that puts the operator given by the matrix
\[ T = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \]
into Jordan canonical form. What is the Jordan canonical form?

Take $v_1 = Av_2 = (0, 0, 1)$, $v_2 = Av_3 = (0, 1, 1)$, $v_3 = (1, 0, 0)$.

Jordan form:
\[ T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \]
6. Consider $\mathbb{R}^2$ with the inner product
\[
\langle (x_1, x_2), (y_1, y_2) \rangle = 2x_1y_1 + x_1y_2 + x_2y_1 + x_2y_2
\]
a. Find an orthonormal basis for $\mathbb{R}^2$ with respect to the above inner product.
Take $e_1 = (0, 1), e_2 = (1, -1)$.

b. Find the vector $v = (a, b)$ closest to $(1, 0)$ satisfying $a + b = 0$.
$v = (1, -1) = \langle (1, 0), e_2 \rangle e_2$. 
7. Find the Jordan form of an operator \( T : \mathbb{C}^5 \to \mathbb{C}^5 \) given the following information:

\[
\dim \text{null}(T^2) = 2 \quad \dim \text{null}(T^3) = 3 \quad \dim \text{null}((T-1)^2) = 2 \quad \dim \text{range}(T-1) = 4
\]

Be sure to justify your answer.

Jordan form:

\[
T = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

Since \( \dim \text{null}(T^3) = 3 \), we have \( \dim \tilde{E}_0 \geq 3 \). Since \( \dim \text{null}((T-1)^2) = 2 \), we have \( \dim \tilde{E}_1 \geq 2 \). Thus since \( \dim \mathbb{C}^5 = 5 \), we must have \( \dim \tilde{E}_0 = 3 \) and \( \dim \tilde{E}_1 = 2 \).

Next, since \( \dim \text{null}(T^2) = 2 \), we must have the Jordan block

\[
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}
\]

Since \( \dim \text{range}(T-1) = 4 \), we can not have \( \dim \text{null}(T-1) = 2 \), and so we must have the Jordan block

\[
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}
\]
8. Consider the following matrices:

\[
T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
T_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad T_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad T_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}
\]

Which of the matrices has minimal polynomial \( m(z) = z^3 + z \)? Be sure to justify your answer.

\( T_2, T_3, T_6 \). They each satisfy \( m(z) = z^3 + z = z(z - i)(z + i) = 0 \) and have each eigenvalue so no factor can be removed.

\( i, -i \) are not eigenvalues of \( T_1, T_4 \).

\( T_5 \) has a Jordan block

\[
\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
\]

which means its minimal polynomial must contain \( z^2 \) as a factor.
9. Consider the matrix

\[ T = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \]

Calculate \( T^{100} \) applied to the vector \((3, 2)\).

\( T \) has eigenvectors \((1, 1), (1, -1)\) with respective eigenvalues 0, 2.

\( (3, 2) = \frac{5}{2} (1, 1) + \frac{1}{2} (1, -1) \).

Thus \( T^{100} (3, 2) = T^{100} (\frac{5}{2} (1, 1) + \frac{1}{2} (1, -1)) = T^{100} (\frac{1}{2} (1, -1)) = 2^{99} (1, -1) \).
10. Let $V$ be a complex vector space of dimension $n$. Suppose $T : V \rightarrow V$ satisfies $T^n = 0$ but $T^{n-1} \neq 0$. Show that there is a vector $v \in V$ such that the list $v, Tv, T^2 v, \ldots, T^{n-1} v$ is a basis.

Since $T^{n-1} \neq 0$, there exists a vector $v \in V$ such that $T^{n-1} v \neq 0$.

Suppose $a_1 v + a_2 Tv + \cdots + a_n T^{n-1} v = 0$. Apply $T^{n-1}$ to obtain $a_1 T^{n-1} v = 0$. Thus $a_1 = 0$ and so $a_2 Tv + \cdots + a_n T^{n-1} v = 0$.

Apply $T^{n-2}$ to obtain $a_2 T^{n-1} v = 0$. Thus $a_2 = 0$ and so $a_3 T^2 v + \cdots + a_n T^{n-1} v = 0$.

Keep repeating to conclude $a_1 = \cdots = a_n = 0$.

Thus $v, Tv, T^2 v, \ldots, T^{n-1} v$ is linearly independent. Since it has size $n = \dim V$, it also must span and hence be a basis.