This exam consists of 10 problems, each worth 10 points, of which you must complete 8. **Choose two problems not to be graded by crossing them out in the box below.** You must justify every one of your answers unless otherwise directed.

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1. Let $V$ be a nonzero finite-dimensional real vector space. Suppose $T : V \to V$ is a linear transformation.

Decide if the following assertions are ALWAYS TRUE or SOMETIMES FALSE. You need not justify your answer.

i. There exists an eigenvalue of $T$.

ii. There exists a basis of $V$ such that $T$ is upper-triangular.

iii. $\dim V = \dim \text{null}(T) + \dim \text{range}(T)$

iv. If $v$ and $w$ are colinear, then $Tv$ and $Tw$ are colinear.

v. If $v$ and $w$ are linearly independent, then $Tv$ and $Tw$ are linearly independent.

vi. If $T$ is invertible and $\lambda$ is an eigenvalue of $T$, then $\lambda^{-1}$ is an eigenvalue of $T^{-1}$.

vii. If $T$ is invertible and $v$ is an eigenvector of $T$, then $v$ is an eigenvector of $T^{-1}$.

viii. If $T^2 = 1$, then $T$ has an eigenvalue.

ix. If $T^3 = T^2$, then $T$ has an eigenvalue.

x. If $T^3 = T^2$, then null($T$) $\neq \{0\}$.
2. Let $V$ be an inner product space and $v_1, \ldots, v_n$ a list of vectors in $V$.

(a) State what it means for $v_1, \ldots, v_n$ to be linearly independent. State what it means for $v_1, \ldots, v_n$ to be orthonormal.

(b) Prove that if $v_1, \ldots, v_n$ is orthonormal, then $v_1, \ldots, v_n$ is linearly independent.
3. Let $A \in M_{n \times n}(\mathbb{C})$ be a complex matrix. Consider the subspace $W \subset M_{n \times n}(\mathbb{C})$ given by

$$W = \text{span}\{I, A, A^2, A^3, \ldots, A^k, \ldots\}$$

Prove that

$$\dim W \leq n.$$
4. Consider $\mathbb{C}^3$ with the standard Euclidean inner product. Determine whether each of the following operators $T : \mathbb{C}^3 \to \mathbb{C}^3$ is self-adjoint, normal, or neither. You need not justify your answer.

   a. $T$ has eigenvectors $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ with respective eigenvalues $0, 1 + i, 1 - i$.

   b. $T$ has eigenvectors $(1, i, 0), (1, -i, 0), (0, 0, 1)$ with respective eigenvalues $1, -1, 0$.

   c. $T$ has eigenvectors $(1, 0, 0), (0, i, -i), (1, 1, 1)$ with respective eigenvalues $1, -1, 1$.

   d. $\dim \text{null}(T^2) = 3, \dim \text{range}(T) = 1$.

   e. $\dim \text{null}(T - i) = 2, \dim \text{null}(T) = 1$ with $\text{null}(T - i) \perp \text{null}(T)$. 
5. Find a basis for $\mathbb{C}^3$ that puts the operator given by the matrix

$$T = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

into Jordan canonical form. What is the Jordan canonical form?
6. Consider $\mathbb{R}^2$ with the inner product

$$\langle (x_1, x_2), (y_1, y_2) \rangle = 2x_1y_1 + x_1y_2 + x_2y_1 + x_2y_2$$

a. Find an orthonormal basis for $\mathbb{R}^2$ with respect to the above inner product.

b. Find the vector $v = (a, b)$ closest to $(1, 0)$ satisfying $a + b = 0$. 
7. Find the Jordan form of an operator \( T : \mathbb{C}^5 \rightarrow \mathbb{C}^5 \) given the following information:

\[
\dim \text{null}(T^2) = 2 \quad \dim \text{null}(T^3) = 3 \quad \dim \text{null}((T-1)^2) = 2 \quad \dim \text{range}(T-1) = 4
\]

Be sure to justify your answer.
8. Consider the following matrices:

\[
T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\]

\[
T_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad T_5 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad T_6 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}.
\]

Which of the matrices has minimal polynomial \( m(z) = z^3 + z? \) Be sure to justify your answer.
9. Consider the matrix

$$T = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

Calculate $T^{100}$ applied to the vector (3, 2).
10. Let $V$ be a complex vector space of dimension $n$. Suppose $T : V \to V$ satisfies $T^n = 0$ but $T^{n-1} \neq 0$. Show that there is a vector $v \in V$ such that the list $v,Tv,T^2v,\ldots,T^{n-1}v$ is a basis.