1. Let $P_{\leq 2}(\mathbb{R})$ denote the real vector space of polynomials of degree less than or equal to two. Consider the linear transformation $T : P_{\leq 2}(\mathbb{R}) \to \mathbb{R}^2$ given by

$$T(p(x)) = \begin{pmatrix} p(-1) \\ p(1) \end{pmatrix}$$

a. What is the matrix of T with respect to the basis $1, x, x^2$ of $P_{\leq 2}(\mathbb{R})$ and the standard basis of $\mathbb{R}^2$?

$$\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

b. Find the dimension of the subspace $U \subset P_{\leq 2}(\mathbb{R})$ of polynomials with $p(-1) = p(1) = 0$. Be sure to justify your answer.

This subspace U is just the null space of T. Its dimension is the same as the dimension of the null space of the matrix obtained in A, which is 1. The easiest way to see this is to note that the rank of the matrix is 2, since the first two columns are independent. So the nullity is 3-2=1.

2. Let $v_1, \ldots, v_n$ be a linearly independent list of vectors of V, and let u_1, u_2 be another linearly independent list of vectors of V. Suppose that u_1 and u_2 are each *not* in $\mathit{span}(v_1, \ldots, v_n)$.

Decide if the following assertion is always true or sometimes false. If always true, provide a proof; if sometimes false, provide a counterexample and justify why it is a counterexample.

Assertion: the list $v_1, \ldots, v_n, u_1, u_2$ is linearly independent.

This is sometimes false. For a counterexample, let $V = \mathbb{R}^2$, $n = 1$, $v_1 = (1, 0)$, $u_1 = (0, 1)$, and $u_2 = (1, 1)$. Then all the conditions are satisfied, but v_1, u_1, u_2 is dependent, because it's a list of length three in a two-dimensional space.

3. Let V be a vector space and $U \subset V$ a subspace with $\dim V = n$ and $\dim U = k$.

Let $L \subset L(V, V)$ be the subset of linear transformations $T : V \to V$ such that U is T-invariant.

a. Check that L is a subspace.

0 is in L because every subspace is invariant under the zero map. If S and T are in L, then let $u \in U$. Since $(S + T)u = Su + Tu$, and Su and Tu are both in U, we find that U

is invariant under $S+T$. Similarly, if $c \in \mathbb{F}$, then $(cT)u = c(Tu)$, and since $Tu \in U$ and U is a subspace, $cTu \in U$, so U is invariant under cT. Thus both $S+T$ and cT are in L, so L is a subspace.

b. Calculate $\dim L$.

Pick a basis $(u_1, \ldots, u_k)$ for U and extend it to a basis $B = (u_1, \ldots, u_k, v_1, \ldots, v_{n-k})$ for all of V. In order for U to be invariant under some map T, we must have $Tu_i = *u_1 + \cdots + *u_k$, in other words, the expression for Tu_i does not involve any v_js. Thus U is T-invariant if and only if the matrix of T in this basis has the form

$$\begin{pmatrix} * & \cdots & * & * & \cdots & * \\ \vdots & \cdots & & \vdots & \vdots & * \\ * & \cdots & * & * & \cdots & * \\ 0 & \cdots & 0 & * & \cdots & * \\ \vdots & \vdots & \vdots & * & \cdots & * \\ 0 & \cdots & 0 & * & \cdots & * \end{pmatrix}$$

This is an $n \times n$ matrix with $k(n-k)$ zeroes in the bottom left corner. The dimension of the space of such matrices is $n^2 - k(n-k) = n^2 - kn + k^2$. This is therefore the dimension of L, since each such matrix corresponds to a unique operator in L.

4. Let V be a finite-dimensional nonzero complex vector space. For each of the following, decide if it is possible for a linear transformation $T : V \to V$ to satisfy the stated requirements. If yes, give an example; if no, justify why not.

a. T is injective but not surjective.

It's not possible: if T is injective, then its null space is zero, so the rank-nullity theorem implies that the range of T is V, so T is surjective, too.

b. $null(T) = range(T)$.

This is possible: for example, take the operator on $\mathbb{R}^2$ whose basis with respect to the standard basis is $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ its nullspace and range are both equal to the x-axis.

c. For any basis of V, the corresponding matrix of T is diagonal.

This is possible: for example, take T to be the identity map I. With respect to any basis, the matrix for I is the identity matrix.

d. There exists a linear transformation $S : V \to V$ such that $ST = \mathrm{Id}_V$ and $TS = 0$.

This is not possible: If $ST = \mathrm{Id}_V$, then T must be injective and S must be surjective, so they're both invertible. But the composition of two invertible maps must be invertible, so TS cannot be zero.

5. Let V be a two-dimensional complex vector space, and $T : V \to V$ a linear transformation satisfying $T^4 = -T^2$.

a. What are the three possible eigenvalues of T?

Suppose that λ is an eigenvalue of T, with a nonzero eigenvector v. Then $\lambda^4 v = T^4 4 = -T^2 v = -\lambda^2 v$ and since $v \neq 0$, we get $\lambda^4 = -\lambda^2$, which implies that λ can only be 0 or $\pm i$.

b. Is it possible for one such linear transformation T to have all three possible eigenvalues? Be sure to justify your answer.

No, it is not possible, for if it were, we would have three independent eigenvectors for T, since eigenvectors associated to different eigenvalues are independent. But it's not possible to have three independent vectors in a two-dimensional space.

6. Consider a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$. Prove that if $null(T) \cap range(T) = \{0\}$ and $\dim range(T) = 3$, then T has at least 2 distinct eigenvalues.

First of all, the rank-nullity theorem implies that $\dim null(T) = 1$, so 0 is an eigenvalue of T. Also, $Range(T)$ is invariant under T (this is true for any operator), so T restricts to an operator on $Range(T)$. Since this is a three-dimensional space, T has a (real) eigenvalue on $Range(T)$, by some theorem from class which says that operators on an odd-dimensional real space must have at least one real eigenvalue. Moreover, this eigenvalue of T on $Range(T)$ cannot be 0, since $Range(T) \cap Null(T) = \{0\}$. So T has two distinct eigenvalues: one zero and the other nonzero.