6-9 Using the angle addition formulas for sine and cosine, we get the following relations, for any integers \( m \) and \( n \):

\[
\sin(mx) \cos(nx) = \frac{1}{2} \left( \sin((m-n)x + (m+n)x) \right)
\]
\[
\sin(mx) \sin(nx) = \frac{1}{2} \left( \cos((m-n)x) - 
\cos((m+n)x) \right)
\]
\[
\cos(mx) \cos(nx) = \frac{1}{2} \left( \cos((m-n)x) + \cos((m+n)x) \right).
\]

The integral of \( \cos(kx) \) or \( \sin(kx) \) is 0 when integrated over a full period \([-\pi, \pi]\), so this means that

\[
\int_{-\pi}^{\pi} \sin(mx) \cos(nx) = 0 \text{ for any } m, n
\]
\[
\int_{-\pi}^{\pi} \sin(mx) \sin(nx) = 0 \text{ for } m \neq n
\]
\[
\int_{-\pi}^{\pi} \cos(mx) \cos(nx) = 0 \text{ for } m \neq n.
\]

The only remaining case is when \( m = n \), and then we get

\[
\int_{-\pi}^{\pi} \sin(mx) \sin(nx) = \int_{-\pi}^{\pi} \frac{1}{2} = \pi,
\]
\[
\int_{-\pi}^{\pi} \cos(mx) \cos(nx) = \int_{-\pi}^{\pi} \frac{1}{2} = \pi.
\]

6-10 First we find an orthogonal basis \( \{b_1, b_2, b_3\} \). We set \( b_1 = 1 \). Then we set

\[
b_2 = x - \frac{\langle x, b_1 \rangle}{\langle b_1, b_1 \rangle} b_1 = x - \frac{1}{2},
\]
\[
b_3 = x^2 - \frac{\langle x^2, b_2 \rangle}{\langle b_2, b_2 \rangle} b_2 - \frac{\langle x^2, b_1 \rangle}{\langle b_1, b_1 \rangle} b_1 = x^2 - (x - \frac{1}{2}) - \frac{1}{3}.
\]

Normalizing, we get

\[
e_1 = 1,
\]
\[
e_2 = \frac{1}{\sqrt{12}} \left( x - \frac{1}{2} \right)
\]
\[
e_3 = \sqrt{\frac{180}{61}} \left( x^2 - x + \frac{1}{6} \right).
\]

6-11 If \( v_n \) is in \( \text{Span}(v_1, \ldots, v_{n-1}) \) and \( v_1, \ldots, v_{n-1} \) are linearly independent, then applying the Gram-Schmidt procedure will give orthonormal vectors \( e_1, \ldots, e_{n-1} \) but spit out \( e_n = 0 \).

6-12 We prove it by induction on \( m \). In case \( m = 1 \), \( \text{Span}(v_1) \) is one-dimensional and there are exactly two vectors with unit norm, namely \( \pm e_1 \) where \( e_1 \) is a vector of unit norm. Now suppose we have found that there are \( 2^{m-1} \) orthonormal lists with \( \text{Span}(v_1, \ldots, v_j) = \text{Span}(e_1, \ldots, e_j) \) for
$j \leq m - 1$. Let $U = \text{Span}(v_1, ..., v_{m-1})$. To obtain an orthonormal basis for $V$, we must start with an orthonormal basis for $U$ and then extend it to a basis for $V$. The last basis vector must be in $U^\perp$, which is 1-dimensional. As explained above there are two vectors of unit norm in $U^\perp$. We see that for each choice of basis for $U$, there are exactly two ways to complete this basis to a basis of $V$ with the desired property. Therefore there are $2^m$ possible choices for bases as we wanted to show.

6-13 Extend $(e_1, ..., e_m)$ to an orthonormal basis $(e_1, ..., e_n)$ of $V$. Then we have $v = \langle v, e_1 \rangle e_1 + ... + \langle v, e_m \rangle e_m$. The vector $v$ lies in $\text{Span}(e_1, ..., e_m)$ if and only if $\langle v, e_i \rangle = 0$ for $m < i \leq n$. By the pythagorean theorem, this happens if and only if $\|v\|^2 = |\langle v, e_1 \rangle|^2 + ... + |\langle v, e_m \rangle|^2$.

6-14 Notice that the differentiation operator has an upper triangular matrix with respect to the standard basis $\{1, x, x^2\}$. Therefore it also has an upper-triangular matrix with respect to the basis obtained from Gram-Schmidt applied to $\{1, x, x^2\}$; see the proof of Corollary 6.27. We computed this basis in Problem 10.

6-15 Use the Rank-Nullity theorem and Theorem 6.29.

6-16 Obvious from Problem 6-15.

6-17 In Problem 5-21, we showed that if $P^2 = P$, then $V = \text{Null}(P) \oplus \text{Range}(P)$. Therefore, the action of $P$ is always as follows: for $v \in V$, write $v = u + w$, for $w \in \text{Null}(P)$ and $u \in \text{Range}(P)$. Then $P(v) = u$. For $P$ to be an orthogonal projection, then, is equivalent to saying that $\text{Null}(P) = \text{Range}(P)^\perp$, which is exactly the problem statement.

6-18 Again, we know that $P$ is the projection onto $\text{Range}(P)$ with kernel $\text{Null}(P)$, by Problem 5-21. We have to show that the condition in the problem statement implies that $\text{Null}(P) = \text{Range}(P)^\perp$. Suppose to the contrary that we have vectors $v \in \text{Null}(P)$, $w \in \text{Range}(P)$, with $\langle v, w \rangle \neq 0$. According to Problem 6-2, there exists a scalar $a \in F$ such that $\|w\| > \|w + av\|$. Applying $P$ on the right-hand side, notice that $P(w + av) = P(w) = w$. Therefore $u = w + av$ is a vector with the property that $\|P(u)\| > \|u\|$, contradicting the problem statement.

6-19 First suppose that $U$ is invariant under $T$, so that $T(U) \subseteq U$. Write $P$ for $P_U$ to ease notation. For any $v \in V$, we want to show that $PTP(v) = TP(v)$. Well, $P(v) \in U$ by definition, so $TP(v) \in U$ by assumption. But $P$ is the identity when restriction to $U$ since it’s a projection, so $PTP(v) = TP(v)$.

Conversely, suppose that $PTP(v) = TP(v)$ for all $v \in V$. Now take any $u \in U$. Then $P(u) = u$, so by assumption, $PT(u) = T(u)$. But for any vector $v \in V$, $P(v) = v$ if and only if $v \in V$ (since $P$ is a projection operator), so the equation $PT(u) = T(u)$ implies that $T(u) \in U$, so $U$ is $T$-invariant.

6-20 First suppose that $U$ and $U^\perp$ are both invariant under $T$. Now take any vector $v \in V$; we want to show that $PT(v) = TP(v)$ (writing $P$ for $P_U$ again). Write $v = u + w$ with $u \in U$ and $w \in U^\perp$. Then $P(v) = u$, and so $TP(v) = T(u)$. On the other hand, $T(u + w) = T(u) + T(w)$ with $T(u) \in U$ and $T(w) \in U^\perp$ by assumption, so $PT(u + w) = PT(u) + PT(w) = T(u)$, since $P$ is the identity on $U$ and is the zero map on $U^\perp$. Therefore $PT(v) = TP(v)$ as we needed to show.

Conversely, suppose that $PT(v) = TP(v)$ for all $v \in V$. We need to show that $U$ and $U^\perp$ are both $T$-invariant. First we show that $U$ is $T$-invariant, so let $u \in U$. Then $P(u) = u$, so $TP(u) = T(u)$. On the other hand $PT(u) = T(u)$ if and only if $T(u) \in U$ since $P$ is the projection onto $U$. Therefore $U$ is $T$-invariant.
Next we show that $U^\perp$ is $T$-invariant. Take any $w \in U^\perp$. Then $P(w) = 0$, and therefore $TP(w) = PT(w) = 0$. But for any vector $v \in V$, $Pv = 0$ if and only if $v \in U^\perp$, so the equation $PT(w) = 0$ implies that $T(w) \in U^\perp$, and therefore $U^\perp$ is $T$-invariant.