MATH 110 HOMEWORK 9 SOLUTIONS

6-9 Using the angle addition formulas for sine and cosine, we get the following relations, for any integers m and n:

$$\sin(mx)\cos(nx) = \frac{1}{2}\Big(\sin(m-n)x + \sin(m+n)x)\Big)$$
$$\sin(mx)\sin(nx) = \frac{1}{2}\Big(\cos(m-n)x - \cos(m+n)x)\Big)$$
$$\cos(mx)\cos(nx) = \frac{1}{2}\Big(\cos(m-n)x + \cos(m+n)x)\Big).$$

The integral of $\cos(kx)$ or $\sin(kx)$ is 0 when integrated over a full period $[-\pi,\pi]$, so this means that

$$\int_{-\pi}^{\pi} \sin(mx)\cos(nx) = 0 \ \text{ for any } m, n$$
$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx) = 0 \ \text{ for } m \neq n$$
$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx) = 0 \ \text{ for } m \neq n.$$

The only remaining case is when $m = n$, and then we get

$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx) = \int_{-\pi}^{\pi} \frac{1}{2} = \pi,$$
$$\int_{-\pi}^{\pi} \cos(mx) cos(mx) = \int_{-\pi}^{\pi} \frac{1}{2} = \pi.$$

6-10 First we find an orthogonal basis $\{b_1, b_2, b_3\}$. We set $b_1 = 1$. Then we set

$$b_2 = x - \frac{\langle x, b_1\rangle}{\langle b_1, b_1\rangle} b_1 = x - 1/2,$$
$$b_3 = x^2 - \frac{\langle x^2, b_2\rangle}{\langle b_2, b_2\rangle} b_2 - \frac{\langle x^2, b_1\rangle}{\langle b_1, b_1\rangle} b_1 = x^2 - (x - \frac{1}{2}) - \frac{1}{3}.$$

Normalizing, we get

$$e_1 = 1,$$
$$e_2 = \frac{1}{\sqrt{12}}\Big(x - 1/2\Big)$$
$$e_3 = \sqrt{\frac{180}{61}}\Big(x^2 - x + 1/6\Big).$$

6-11 If v_n is in $\text{Span}(v_1, ..., v_{n-1})$ and $v_1, ..., v_{n-1}$ are linearly independent, then applying the Gram-Schmidt procedure will give orthonormal vectors $e_1, ..., e_{n-1}$ but spit out $e_n = 0$.

6-12 We prove it by induction on m. In case $m = 1$, $\text{Span}(v_1)$ is one-dimensional and there are exactly two vectors with unit norm, namely $\pm e_1$ where e_1 is a vector of unit norm. Now suppose we have found that there are 2^{m-1} orthonormal lists with $\text{Span}(v_1, ..., v_j) = \text{Span}(e_1, ...e_j)$ for

$j \leq m-1$. Let $U = \text{Span}(v_1, ..., v_{m-1})$. To obtain an orthonormal basis for V, we must start with an orthonormal basis for U and then extend it to a basis for V. The last basis vector must be in $U^\perp$, which is 1-dimensional. As explained above there are two vectors of unit norm in $U^\perp$. We see that for each choice of basis for U, there are exactly two ways to complete this basis to a basis of V with the desired property. Therefore there are 2^m possible choices for bases as we wanted to show.

6-13 Extend $(e_1, ..., e_m)$ to an orthonormal basis $(e_1, ..., e_n)$ of V. Then we have $v = \langle v, e_1\rangle e_1 + ... + \langle v, e_n\rangle e_n$. The vector v lies in $\text{Span}(e_1, ..., e_m)$ if and only if $\langle v, e_i\rangle = 0$ for $m < i \leq n$. By the pythagorean theorem, this happens if and only if $||v||^2 = |\langle v, e_1\rangle|^2 + ... + |\langle v, e_m\rangle|^2$.

6-14 Notice that the differentiation operator has an upper triangular matrix with respect to the standard basis $\{1, x, x^2\}$. Therefore it also has an upper-triangular matrix with respect to the basis obtained from Gram-Schmidt applied to $\{1, x, x^2\}$; see the proof of Corollary 6.27. We computed this basis in Problem 10.

6-15 Use the Rank-Nullity theorem and Theorem 6.29.

6-16 Obvious from Problem 6-15.

6-17 In Problem 5-21, we showed that if $P^2 = P$, then $V = \text{Null}(P) \oplus \text{Range}(P)$. Therefore, the action of P is always as follows: for $v \in V$, write $v = u + w$, for $w \in \text{Null}(P)$ and $u \in \text{Range}(P)$. Then $P(v) = u$. For P to be an orthogonal projection, then, is equivalent to saying that $\text{Null}(P) = \text{Range}(P)^\perp$, which is exactly the problem statement.

6-18 Again, we know that P is the projection onto $\text{Range}(P)$ with kernel $\text{Null}(P)$, by Problem 5-21. We have to show that the condition in the problem statement implies that $\text{Null}(P) = \text{Range}(P)^\perp$. Suppose to the contrary that we have vectors $v \in \text{Null}(P)$, $w \in \text{Range}(P)$, with $\langle v, w\rangle \neq 0$. According to Problem 6-2, there exists a scalar $a \in F$ such that $||w|| > ||w + av||$. Applying P on the right-hand side, notice that $P(w + av) = P(w) = w$. Therefore $u = w + av$ is a vector with the property that $||P(u)|| > ||u||$, contradicting the problem statement.

6-19 First suppose that U is invariant under T, so that $T(U) \subseteq U$. Write P for P_U to ease notation. For any $v \in V$, we want to show that $PTP(v) = TP(v)$. Well, $P(v) \in U$ by definition, so $TP(v) \in U$ by assumption. But P is the identity when restriction to U since it's a projection, so $PTP(v) = TP(v)$.

Conversely, suppose that $PTP(v) = TP(v)$ for all $v \in V$. Now take any $u \in U$. Then $P(u) = u$, so by assumption, $PT(u) = T(u)$. But for any vector $v \in V$, $P(v) = v$ if and only if $v \in V$ (since P is a projection operator), so the equation $PT(u) = T(u)$ implies tha $T(u) \in U$, so U is T-invariant.

6-20 First suppose that U and $U^\perp$ are both invariant under T. Now take any vector $v \in V$; we want to show that $PT(v) = TP(v)$ (writing P for P_U again). Write $v = u + w$ with $u \in U$ and $w \in U^\perp$. Then $P(v) = u$, and so $TP(v) = T(u)$. On the other hand, $T(u + w) = T(u) + T(w)$ with $T(u) \in U$ and $T(w) \in U^\perp$ by assumption, so $PT(u + w) = PT(u) + PT(w) = T(u)$, since P is the identity on U and is the zero map on $U^\perp$. Therefore $PT(v) = TP(v)$ as we needed to show.

Conversely, suppose that $PT(v) = TP(v)$ for all $v \in V$. We need to show that U and $U^\perp$ are both T-invariant. First we show that U is T-invariant, so let $u \in U$. Then $P(u) = u$, so $TP(u) = T(u)$. On the other hand $PT(u) = T(u)$ if and only if $T(u) \in U$ since P is the projection onto U. Therefore U is T-invariant.

Next we show that $U^\perp$ is T-invariant. Take any $w \in U^\perp$. Then $P(w) = 0$, and therefore $TP(w) = PT(w) = 0$. But for any vector $v \in V$, $Pv = 0$ if and only if $v \in U^\perp$, so the equation $PT(w) = 0$ implies that $T(w) \in U^\perp$, and therefore $U^\perp$ is T-invariant.