Homework 6 Solutions

1. Suppose $u \in U_1 + \cdots + U_m$. Then for $i = 1, \dots, m$, $\exists u_i \in U_i$ such that $u = u_1 + \cdots + u_m$. Thus, $Tu = Tu_1 + \dots Tu_m$, and by the T-invariance of U_i, $Tu_i \in U_i$ for $i = 1, \dots m$. Therefore, $Tu \in U_1 + \cdots + U_m$ and so $U_1 + \cdots + U_m$ is T-invariant. $\square$

2. Suppose $\{U_x\}_{x \in \Gamma}$ is a collection of T-invariant subspaces. If $u \in \cap_{x\in\Gamma} U_x$, then $u \in U_x$ for all $x \in \Gamma$. By T-invariance of U_x, $Tu \in U_x$ for all $x \in \Gamma$. Therefore, $Tu \in \cap_{x\in\Gamma} U_x$, and so $\cap_{x\in\Gamma} U_x$ is T -invariant. $\square$

3. Claim: If U is a subspace of V invariant under all $T \in \mathcal{L}(V)$ then $U = \{0\}$ or $U = V$. proof: We will prove the contrapositive. Assume $U \neq \{0\}, V$. We will find a $T \in \mathcal{L}(V)$ such that U is not T-invariant. Choose $x \in U \setminus \{0\}$ and $y \notin U$. By the basis extension theorem, we can find a basis $\{x, b_1, \dots, b_n\}$ of V. Then define T by $T(x) = y$ and $T(b_i) = 0$ for $i = 1, \dots n$ and extend by linearity. Then $T \in \mathcal{L}(V)$ and T maps $u \in U$ to an element outside of U. Therefore, U is not invariant under T. $\square$

4. . Suppose $S, T \in \mathcal{L}(V)$, and that $ST = TS$. Let $v \in \text{null}(T - \lambda I)$. Then
$$(T - \lambda I)(Sv) = TSv - \lambda Sv = STv - \lambda Sv = S(Tv - \lambda v) = 0.$$
Thus, $Sv \in \text{null}(T - \lambda I)$. Therefore, $\text{null}(T - \lambda I)$ is S-invariant. $\square$

5. Define $T \in \mathcal{L}(F^2)$ by $T(w, z) = (z, w)$. If λ is an eigenvalue of T, then for some nonzero $(w, z) \in F^2$, $T(w, z) = \lambda(w, z)$, which implies $z = \lambda w$, and $w = \lambda z$. Note that we can assume both $w, z \neq 0$, because if one is 0 then so is the other, which contradicts the assumption that (w, z) is nonzero. By substitution $w = \lambda^2 w$, and since $w \neq 0$, this implies $\lambda^2 = 1$ which means $\lambda = 1$, or -1. Then it follows that $E_1(T) = \{(w, w) : w \in F\}$ and $E_{-1}(T) = \{(w, -w) : w \in F\}$ are the corresponding eigenspaces.

6. Define $T \in \mathcal{L}(F^3)$ by $T(z_1, z_2, z_3) = (2z_2, 0, 5z_3)$. Then solving the system $T(z, y, z) = \lambda(x, y, z)$ shows the eigenvalues are $\lambda = 0, 5$ and the eigenspaces are $E_0 = \text{span}\{e_1\}$, and $E_5 = \text{span}\{e_3\}$

7. Define $T \in \mathcal{L}(F^n)$ by $T(x_1, \dots, x_n) = (x_1 + \dots x_n, \dots, x_1 + \dots x_n)$. Then $Tx = \lambda x$ implies $x_1 + \cdots + x_n = \lambda x_i$ for $i = 1, \dots, n$. Hence, $\lambda = 0$ or $x_1 = \cdots = x_n$. One can check that in the first case $\lambda = 0$, and $E_0(T) = \{(x_1, \dots, x_n) \in F^n : x_1 + \cdots + x_n = 0\}$. In the later case, $\lambda = n$, and $E_n(T) = span\{(1, 1, \dots, 1)\}$.

8. Let $T \in \mathcal{L}(F^\infty)$ by the left shift operator. Solving the equations $z_{i+1} = \lambda z_i$ for $i = 1, 2 \dots$ shows we can let z_1 be free, and then $z_{i+1} = \lambda^i z_1$ for $i = 1, 2, \dots$. So every $\lambda \in F$ is an eigenvalue, and the corresponding eigenspace is given by $E_\lambda(T) = \text{span}\{(1, \lambda, \lambda^2, \dots)\}$.

9. Suppose $T \in \mathcal{L}(V)$ and dim range$T = k$. Let $\lambda_1, \dots, \lambda_m$ denote the distinct eigenvalues of T. Then we need to show that $m \leq k + 1$. Let $\mathcal{S} = \{v_1, \dots, v_m\}$ be a set of corresponding eigenvectors. So $v_i \neq 0$ and $Tv_i = \lambda_i v_i$. In particularly, Then at most one $\lambda_i = 0$. The eigenvectors corresponding to nonzero eigenvalues are in range$T$ since for $\lambda_i \neq$, $T(\frac{1}{\lambda_i} v_i) = v_i$. So $\mathcal{S}$ contains a linearly independent subset of range$T$ that has size at least $m - 1$. Thus, $m - 1 \leq k =$dim range$T$, and so $m \leq k + 1$. $\square$

10. Assume $T \in \mathcal{L}(V)$ is invertible. claim: λ is an eigenvalue of T if and only if λ^{-1} is an eigenvalue of T^{-1}. pf: ($\Longrightarrow$) If λ is an eigenvalue of T, then $\exists v \neq 0$ such that $Tv = \lambda v$. Applying T^{-1} to both sides shows that $v = \lambda T^{-1} v$, which implies $T^{-1} v = \frac{1}{\lambda} v$, which implies λ^{-1} is an eigenvalue of T^{-1}. For the other direction, replace T with T^{-1} and λ with λ^{-1} and use the proof for the $\Longrightarrow$ direction. $\square$

11. Suppose λ is an eigenvalue of ST, and let v denote a λ eigenvector. Then

$$(TS)(Tv) = T(STv) = T(\lambda v) = \lambda Tv \tag{1}$$

if $Tv \neq 0$, then this shows that λ is an eigenvalue of TS. Otherwise, if $Tv = 0$, then $0 = STv = \lambda v$, which implies that $\lambda = 0$ since we know that $v \neq 0$. Then this implies that TS is not invertible, since T is not invertible. Hence $\lambda = 0$ is also an eigenvalue of TS. In either case, we have shown that if λ is an eigenvalue of ST than it is also an eigenvalue of TS. By reversing the roles of S and T we can also conclude that if λ is an eigenvalue of TS than it is also an eigenvalue of ST. □

12. Suppose $T \in \mathcal{L}(V)$ is such that ever $v \in V$ is an eigenvector of T. Then for every $v \in V$ there exists $a_v \in F$ such that $Tv = a_v V$. We need to show that a_v is independent of v for $v \neq 0$. Let $0 \neq w \in V$. Then if $w = cv$ for some $c \in F$ we have that

$$a_w w = Tw = cTv = ca_v v = a_v w \tag{2}$$

which implies $a_v = a_w$. Otherwise, if $w \neq 0$ is not a multiple of v, then $\{v, w\}$ is linearly independent and

$$a_{v+w}(v + w) = T(v + w) = Tv + Tw = a_v v + a_w w \tag{3}$$

which implies that

$$(a_{v+w} - a_v)v + (a_{v+w} - a_w)w = 0 \tag{4}$$

The linear independence of $\{v, w\}$ implies that $a_v = a_{v+w} = a_w$, and so in either case we've shown that the constant is the same for all nonzero $v \in V$. □