① Let $\lambda_1, \dots, \lambda_m \in \mathbb{F}$ be distinct (possible b/c of Axler's running assumption that $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$), and set

$$p(z) := \prod_{j=1}^{m} (z - \lambda_j)$$

② Let $z_1, \dots, z_m \in \mathbb{F}$ be distinct, and define $T : \mathcal{P}_m(\mathbb{F}) \to \mathbb{F}^{m+1}$ by $Tp := (p(z_1), \dots, p(z_{m+1}))$. We need to show that T is bijective. First, note that T is linear:

$$\begin{aligned} T(ap + bq) &= ((ap + bq)(z_1), \dots, (ap + bq)(z_{m+1})) \\ &= (ap(z_1) + bq(z_1), \dots, ap(z_{m+1}) + bq(z_{m+1})) \\ &= a \cdot Tp + bTq. \end{aligned}$$

By rank-nullity, since $\dim(\mathcal{P}_m(\mathbb{F})) = m+1 = \dim(\mathbb{F}^{m+1})$, T is bijective $\iff$ T is injective. But T is obviously injective: If $Tp = 0$, then p is a polynomial of degree $\leq m$ w/ $\geq m+1$ distinct roots; whence $p = 0$.

③ Suppose $p, r_1, r_2, s_1, s_2 \in \mathcal{P}(\mathbb{F})$ are s.t. $\deg r_1 < \deg p$, $\deg r_2 < \deg p$, and $s_1 p + r_1 = s_2 p + r_2$. Then $(s_1 - s_2)p = r_2 - r_1$, whence

$$\deg((s_1 - s_2)p) = \deg(r_2 - r_1) \leq \max\{\deg r_1, \deg r_2\} < \deg p.$$

But if $s_1 \neq s_2$, $\deg((s_1 - s_2)p) \geq \deg p$. Thus, $s_1 = s_2$, which also forces $r_1 = r_2$.

④ Let $p \in \mathcal{P}(\mathbb{C})$ have degree $m > 0$. Then we may write $p(z) = c \cdot \prod_{j=1}^{m} (z - \lambda_j)$. Invoking the product rule, we see that $p'(z) = c \cdot \sum_{j=1}^{m} \prod_{i \neq j} (z - \lambda_i)$. $\forall k \in \{1, \dots, m\}$, $p'(\lambda_k) = c \cdot \prod_{i \neq k} (\lambda_k - \lambda_i)$, since $\prod_{i \neq j} (\lambda_k - \lambda_i) = 0$ for $j \neq k$. Since $\lambda_1, \dots, \lambda_m$ are all the roots of p, we see that p and p' have a root in common

$\iff \exists k \in \{1, \dots, m\}$ s.t. $c \cdot \prod_{i \neq k} (\lambda_k - \lambda_i) = p'(\lambda_k) = 0$

$\iff \exists i, k \in \{1, \dots, m\}$ s.t. $i \neq k$ and $\lambda_i = \lambda_k$

$\iff$ the roots of p are not distinct.

⑤ Suppose $p \in \mathcal{P}(\mathbb{R})$ has no real roots. By Thm. 4.14, we may write p as a product of irreducible quadratic factors, i.e., $p(x) = c \cdot \prod_{j=1}^{M} (x^2 + \alpha_j x + \beta_j)$ where each $\alpha_j^2 < 4\beta_j$. But then $\deg p = 2M$, which is even.

(A.P.) ① Since $\mathbb{R} \subseteq \mathbb{C}$ and all the vector space axioms hold for all complex numbers, they hold for all real numbers as well.

② Each $v \in V$ may be written uniquely as $v = c_1 v_1 + \dots + c_n v_n$, where $c_1, \dots, c_n \in \mathbb{C}$. Writing each $c_j = a_j + b_j \cdot i$ ($a_j, b_j \in \mathbb{R}$), we have that $v = a_1 v_1 + b_1 (i v_1) + \dots + a_n v_n + b_n (i v_n)$, a linear comb. w/ real coefficients of $v_1, iv_1, \dots, v_n, iv_n$. Suppose $a_1 v_1 + b_1 (iv_1) + \dots + a_n v_n + b_n (i v_n) = 0$. Then $(a_1 + b_1 i) v_1 + \dots + (a_n + b_n i) v_n = 0$, whence each $a_j + b_j i = 0$, whence each $a_j = 0 = b_j$.

③ Consider the basis $(v_1, iv_1, \dots, v_n, iv_n)$ of V, and $\forall k \in \{1, \dots, 2k\}$, let e_k denote the k^{th} standard basis vector. Since $Tv_j = v_j + iv_j$, its coordinates w/r.t. the above basis are $e_{2j-1} + e_{2j}$, and since $T(iv_j) = -v_j + iv_j$, its coordinates are $-e_{2j-1} + e_{2j}$. Thus, the matrix of T w/r.t. the above basis is the $(2n) \times (2n)$ matrix whose $(2j-1)^{th}$ column is $e_{2j-1} + e_{2j}$ and whose $(2j)^{th}$ column is $-e_{2j-1} + e_{2j}$.

In other words

$$\begin{bmatrix} 1 & -1 & & & & & \\ 1 & 1 & & & & 0 & \\ & & 1 & -1 & & & \\ & & 1 & 1 & & & \\ & & & & \ddots & & \\ & 0 & & & & 1 & -1 \\ & & & & & 1 & 1 \end{bmatrix}$$