Let $\lambda_1, \ldots, \lambda_m \in F$ be distinct (possible b/c of Artin’s running assumption that $F = \mathbb{R}$ or $\mathbb{C}$), and set $p(z) = \prod_{j=1}^m (z - \lambda_j)$

Let $z_1, \ldots, z_m \in F$ be distinct, and define $T : P_m(F) \to F^{m+1}$ by $T_p := (p(z_1), \ldots, p(z_{m+1}))$. We need to show that $T$ is bijective. First, note that $T$ is linear:

$$T(\alpha p + \beta q) = ((\alpha p + \beta q)(z_1), \ldots, (\alpha p + \beta q)(z_{m+1}))$$

$$= (\alpha p(z_1) + \beta q(z_1), \ldots, \alpha p(z_{m+1}) + \beta q(z_{m+1}))$$

$$= \alpha Tp + \beta Tq.$$ 

By rank-nullity, since $\dim(P_m(F)) = m+1 = \dim(F^{m+1})$, $T$ is bijective $\iff$ $T$ is injective. But $T$ is obviously injective: If $Tp = 0$, then $p$ is a polynomial of degree $\leq m$ with $m+1$ distinct roots, whence $p = 0$.

Suppose $p, r_1, r_2, s_1, s_2 \in P_m(F)$ are st. $\deg r_1 < \deg p$, $\deg r_2 < \deg p$, and $s_1 p + r_1 = s_2 p + r_2$. Then $(s_1 - s_2)p = r_2 - r_1$, whence

$$\deg((s_1 - s_2)p) = \deg(r_2 - r_1) \leq \max\{\deg r_1, \deg r_2\} < \deg p.$$ 

But if $s_1 \neq s_2$, $\deg((s_1 - s_2)p) \geq \deg p$. Thus, $s_1 = s_2$, which also forces $r_1 = r_2$. 
(4) Let \( p \in \mathbb{P}(\mathbb{C}) \) have degree \( m > 0 \). Then we may write
\[
p(z) = c \cdot \prod_{j=1}^{m} (z - \lambda_j).
\]
Invoking the product rule, we see that
\[
p'(z) = c \cdot \prod_{j=1}^{m} (z - \lambda_j), \quad \forall k \in \{1, \ldots, m\}, \quad p'(\lambda_k) = c \cdot \prod_{j \neq k} (\lambda_k - \lambda_j),
\]
since \( \prod_{j \neq k} (\lambda_k - \lambda_j) = 0 \) for \( j \neq k \). Since \( \lambda_1, \ldots, \lambda_m \) are all the roots of \( p \), we see that \( p \) and \( p' \) have a root in common \( \iff \exists k \in \{1, \ldots, m\} \) st. \( c \cdot \prod_{j \neq k} (\lambda_k - \lambda_j) = p'(\lambda_k) = 0 \Rightarrow \exists i, k \in \{1, \ldots, m\} \) st. \( i \neq k \) and \( \lambda_i = \lambda_k \Rightarrow \) the roots of \( p \) are not distinct.

(5) Suppose \( p \in \mathbb{P}(\mathbb{R}) \) has no real roots. By Thm. 4.14, we may write \( p \) as a product of irreducible quadratic factors, i.e.,
\[
p(x) = c \cdot \prod_{j=1}^{m} (x^2 + a_j x + b_j)
\]
where each \( a_j^2 < 4b_j \).
But then \( \deg p = 2m \), which is even.

A.P. (i) Since \( \mathbb{R} \subseteq \mathbb{C} \) and all the vector space axioms hold for all complex numbers, they hold for all real numbers as well.

(ii) Each \( v \in V \) may be written uniquely as \( v = c_1 v_1 + \ldots + c_n v_n \), where \( c_1, \ldots, c_n \in \mathbb{C} \). Writing each \( c_j = a_j + b_j i \) (\( a_j, b_j \in \mathbb{R} \)), we have that
\[
v = a_1 v_1 + b_1 i v_1 + \ldots + a_n v_n + b_n i v_n, \quad \text{a linear comb. w/ real coefficients of } v_1, i v_1, \ldots, v_n, i v_n.
\]
Suppose
\[
a_1 v_1 + b_1 i v_1 + \ldots + a_n v_n + b_n i v_n = 0.
\]
Then
\[
(a_1 + b_1 i) v_1 + \ldots + (a_n + b_n i) v_n = 0 \Rightarrow \text{whence each } a_j + b_j i = 0, \text{ whence each } a_j = 0 = b_j.
Consider the basis \((v_1, v_2, \ldots, v_n, i_{i,v_j})\) of \(V\), and let \(e_k\) denote the \(k\)th standard basis vector. Since \(Tv_j = v_j + iv_j\), its coordinates w.r.t. the above basis are \(e_{j-1} + e_j\). And since \(T(i_{i,v_j}) = -v_j + iv_j\), its coordinates are \(-e_{j-1} + e_j\). Thus, the matrix of \(T\) w.r.t. the above basis is the \((2n) \times (2n)\) matrix whose \((j-1)\)th column is \(e_{j-1} + e_j\) and whose \(j\)th column is \(-e_{j-1} + e_j\).

In other words

\[
\begin{bmatrix}
1 & -1 & 0 \\
1 & 1 & -1 \\
\vdots & \ddots & \ddots \\
1 & \cdots & 1 & -1 \\
0 & \cdots & 1 & 1
\end{bmatrix}
\]