Solutions to Homework #4.

12. First assume there is a surjective map $T\colon V \to W$, so the range of T is W. Applying Theorem 3.4 gives $\dim V = \dim \operatorname{Null} T + \dim W$, so $\dim W \leq \dim V$. For the other direction, suppose $\dim V = n$, and $\dim W = m$, where $m \leq n$. Pick bases $(v_1, \ldots, v_n)$ for V and $(w_1, \ldots, w_m)$ for W (note we used the finite-dimensionality of V and W here). Then define a map T by sending v_i to w_i for $i = 1, \ldots m$, and all other v_i to zero. This map is surjective, since the range is spanned by the Tv_i (this is true for any map T), and by our construction these include all the w_i, so they span W.

13. First assume there is a map T whose null space is U. Then by Theorem 3.4 we have $\dim V = \dim U + \dim \operatorname{Range} T \leq \dim U + \dim W$, where the inequality comes from the fact that the range is a subspace of W. Rearranging this inequality gives $\dim V - \dim W \leq \dim U$. For the other direction, assume that $\dim V - \dim W \leq \dim U$. Pick a basis $(u_1, \ldots, u_m)$ for U and extend it to a basis $(u_1, \ldots, u_m, v_1, \ldots, v_k)$ for V. Note that $k = \dim V - \dim U \leq \dim W$, by our assumption. Therefore it is possible to pick an independent list $(w_1, \ldots, w_k)$ of length k in W. Define a map $T\colon V \to W$ by setting $Tu_i = 0$ for $i = 1, \ldots, m$, and $Tv_i = w_i$, for $i = 1, \ldots, k$. Then this map has nullspace equal to U. Certainly U is contained in the null space, since each u_i goes to zero. But there can't be anything else in the nullspace either, for if some linear combination $a_1v_1 + \cdots + a_kv_k$ goes to zero under T, then by linearity we would have $a_1w_1 + \cdots + a_kw_k = 0$, which forces all a_i to be zero by independence of the w_i. Thus the null space of this map T is exactly U.

14. First assume that T is injective. We must produce a "left-inverse" to T. We pick a basis $(w_1, \ldots, w_m)$ for Range T, extend it to a basis $(w_1, \ldots, w_m, \ldots, w_n)$ and define $S\colon W \to V$ as follows. Since each of $w_1, \ldots, w_m$ is in the range of T, there exists, for each i, a $v_i \in V$ with $Tv_i = w_i$ ($i = 1, \ldots, m$). Now we define our map S by setting $Sw_1 = v_1, \ldots, Sw_m = v_m$, and $Sw_{m+1} = \cdots = Sw_n = 0$. Now we show that ST is the identity map on V. For this it is sufficient to show that Null $ST = 0$. But Null $ST \subseteq$ Null $T = 0$ since T is injective. For the other direction, assume there is a map S with ST the identity map on V. Suppose $v \in$ Null T. Then $Tv = 0$, so $STv = 0$. But $STv = v$, so v was zero to begin with. This means the null space of T is 0, so T is injective.

15. First assume T is surjective. We produce a "right-inverse" to T. Pick a basis $v_1, \ldots, v_n$ for V. Then $Tv_1, \ldots, Tv_n$ span the range of T, which is all of W by assumption of surjectivity. So we may reduce the list $Tv_1, \ldots, Tv_n$ to a basis for W. After possibly reordering the Tv_is we may assume this basis is $Tv_1, \ldots, Tv_k$. Now we define our map $S\colon W \to V$ using this basis, by setting $S(Tv_i) = v_i$ for $i = 1, \ldots, k$. Then for each basis vector Tv_i, we have $(TS)(Tv_i) = T(STv_i) = Tv_i$ so TS is the identity on this basis, hence TS is the identity map on W. For the other direction, assume there is such a map S. Pick any $w \in W$. Then $w = (TS)w = T(Sw)$, so each w is in the range of T, hence T is surjective.

16. First observe that Null $T \subseteq$ Null ST, since if $Tu = 0$, then also $STu = 0$. By theorem 2.13, we can find a subspace Y of Null ST such that Null $ST =$ Null $T \oplus Y$ (this Y is then also a subspace of U). Pick a basis $(u_1, \ldots, u_k)$ for Y. Then we have $\dim \operatorname{Null} ST = \dim \operatorname{Null} T + k$. Now, the Tu_i are independent, since if $a_1Tu_1 + \cdots + a_kTu_k = 0$, then $a_1u_1 + \cdots + a_ku_k \in$ Null T, but Null $T \cap Y = 0$, so this is impossible (this says informally that T is injective when applied only to Y). Moreover,

the Tu_i are in Null S (since $u_i \in$ Null ST), hence can be extended to a basis of Null T. Thus

$$\dim \text{Null } ST = \dim \text{Null } T + k \leq \dim \text{Null } T + \dim \text{Null } S$$

20. We produce an inverse function S. Given an $n \times 1$ column vector, which typesetting requires me to write as a row, call it $(a_1, \ldots, a_n)$, we define $S(a_1, \ldots, a_n)$ to be the vector $a_1 + \cdots + a_n v_n \in V$. Let us check that ST is the identity map on V (by exercise 23, this also shows that TS is the identity, and hence that S and T are indeed inverses). Pick any v in V and write it as $v = c_1 v_1 + \cdots + c_n v_n$. Then Tv is the "column vector" $(c_1, \ldots, c_n)$, and by our definition above, applying S to this gives us $c_1 v_1 + \cdots + c_n v_n$, so ST is the identity map.

22. First assume that both S and T are invertible, with inverse maps S^{-1} and T^{-1}, respectively. Then $T^{-1}S^{-1}$ is the inverse to ST, since $(ST)(T^{-1}S^{-1}) = SIS^{-1} = I$, and similarly $(T^{-1}S^{-1})(ST) = I$. Conversely, suppose that ST is invertible. Then it is both surjective and injective. Since it's injective, Null $ST = 0$. But Null $T \subseteq$ Null $ST = 0$, so T is injective also. By theorem 3.21, this means T is invertible. Similarly, since ST is surjective, Range $ST = V$. But Range $S \supseteq$ Range $ST = V$, so S is surjective, hence invertible. Thus both T and S are invertible.

23. Assume that $ST = I$. Since I is invertible, ST is invertible, so both T and S are surjective and injective, by the previous problem. To check that $TS = I$, we pick any $v \in V$ and show that $TSv = v$. But by surjectivity of T, $v = Tu$ for some $u \in V$, so $TSv = TSTu = TIu = Tu = v$, which is what we wanted. The other direction is the same - just swap S and T.

24. If $T = cI$, then for any S, and any $v \in V$, $STv = S(cv) = cSv$, while $TSv = cI(Sv) = cSv$, so since v was arbitrary, $ST = TS$. The other direction is the hard part. So pick a map T, which has the property that $ST = TS$ for every map $S \in \mathcal{L}(V)$. We're going to apply this assumption to a few special maps. First choose a basis $(v_1, \ldots, v_n)$ for V, and define maps $\phi_{ij} \colon V \to V$ by

$$\phi_{ij}(v_k) = \begin{cases} v_j & \text{if } k = i \\ 0 & \text{if } k \neq i \end{cases}$$

So, for example. the map ϕ_{23} sends v_2 to v_3 and kills all the other basis vectors. This is the abstract/linear map version of the "elementary matrices" E_{ij}, which you might have seen before. Now by our assumption on T, it commutes with all these ϕs, i.e., $T\phi_{ij} = \phi_{ij}T$ for all i, j. Now, we want to know what T does to each v_i, so pick one of them. We'll compute $\phi_{ij}Tv_i$ and $T\phi_{ij}v_i$ and set them equal to one another, by our assumption. Firstly, write $Tv_i = a_1 v_1 + \cdots + a_n v_n$. Then apply ϕ_{ij}: $\phi_{ij}Tv_i = \phi_{ij}(a_1 v_1 + \cdots + a_n v_n) = a_i v_j$, because ϕ_{ij} kills all the vs except v_i, which it sends to v_j. On the other hand, we compute $T\phi_{ij}v_i = Tv_j$. Setting them equal to one another shows that $Tv_j = a_i v_j$. This is true for each j, so we've found that T just scales each basis vector (in terms of matrices, this would mean the matrix for T in this basis is diagonal). But now thinking of j as fixed, and varying i, we see that the equation $Tv_j = a_i v_j$ forces all the a_is to be the same (because the left hand side doesn't involve i at all!). So $a_1 = \cdots = a_n = c$, for some scalar c. Thus we've found that $Tv_j = cv_j$ for each v_j. So T is just scaling by c on the basis vectors. By linearity, T is just scaling by c on *all* vectors, so $T = cI$.

25. The subset of noninvertible operators in $\mathcal{L}(V)$ is not closed under addition. For instance, take the maps ϕ_{ii} of the previous problem (i.e., just those ϕ_{ij} where $i = j$). Certainly each

ϕ_{ii} is not invertible (it kills all the other v_j, so it has $n-1$-dimensional null space). However, $\phi_{11}+\phi_{22}+\cdots+\phi_{nn}$ is the identity map, which is no longer in the set of noninvertible maps.

26. Notice that the first system of equations can be written as $Ax=0$, where A is the $n \times n$ matrix whose i,j entry is the coefficient a_{ij} in the system of equations, and $x=(x_1,\ldots,x_n) \in \mathbb{F}^n$, and 0 means the zero vector in $\mathbb{F}^n$. Similarly, the system of equations in (b) can be written as $Ax=c$, where $c=(c_1,\ldots,c_n) \in \mathbb{F}^n$. Multiplication by A is a linear map $\mathbb{F}^n \to \mathbb{F}^n$, so (a) is equivalent to saying that the linear map is injective. On the other hand, the condition in (b) (for $Ax=c$ to have a solution for every c) is equivalent to saying that multiplication by A is surjective. Bu this linear map is an operator on $\mathbb{F}^n$, so by 3.21 its injectivity and surjectivity are equivalent.

Additional Problem: Determine exactly which 2×2 real matrices give rise to invertible maps $\mathbb{R}^2 \to \mathbb{R}^2$. Solution: Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be our matrix. We want to give conditions on a,b,c,d that ensure invertibility. The condition is that $ad-bc \neq 0$. Let's prove that... First, it is true in general that a map is invertible if and only if, when applied to a basis of the domain, it yields a basis for the codomain (reason: you can define the inverse map by simply sending the codomain basis back to the original basis). In our case, take the standard basis e_1, e_2 for $\mathbb{R}^2$. Then multiplying by A gives two new vectors $\begin{pmatrix} a \\ c \end{pmatrix}$ and $\begin{pmatrix} b \\ d \end{pmatrix}$. So by the discussion above, we will have an isomorphism precisely when these two columns of A are independent. Let's investigate their independence... They're independent if and only if the equation

$$\alpha \begin{pmatrix} a \\ c \end{pmatrix} + \beta \begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

has a solution with one or both of α, β nonzero; without loss of generality we can consider whether β is zero or not (since we may assume neither column is zero). The equation above is equivalent to the system

$$\begin{aligned} \alpha a + \beta b &= 0 \\ \alpha c + \beta d &= 0. \end{aligned}$$

Multiply the first equation by c and the second by a and subtract, giving

$$\beta ad - \beta bc = \beta(ad-bc) = 0$$

Thus the system has a nontrivial solution with β nonzero if and only if $ad-bc=0$. Equivalently, the columns of A are independent if and only if $ad-bc \neq 0$.