Solutions to Homework #4.

12. First assume there is a surjective map $T: V \to W$, so the range of $T$ is $W$. Applying Theorem 3.4 gives $\dim V = \dim \text{Null} T + \dim W$, so $\dim W \leq \dim V$. For the other direction, suppose $\dim V = n$, and $\dim W = m$, where $m \leq n$. Pick bases $(v_1, \ldots, v_n)$ for $V$ and $(w_1, \ldots, w_m)$ for $W$ (note we used the finite-dimensionality of $V$ and $W$ here). Then define a map $T$ by sending $v_i$ to $w_i$ for $i = 1, \ldots, m$, and all other $v_i$ to zero. This map is surjective, since the range is spanned by the $Tv_i$ (this is true for any map $T$), and by our construction these include all the $w_i$, so they span $W$.

13. First assume there is a map $T$ whose null space is $U$. Then by Theorem 3.4 we have $\dim V = \dim U + \dim \text{Range} T \leq \dim U + \dim W$, where the inequality comes from the fact that the range is a subspace of $W$. Rearranging this inequality gives $\dim V - \dim W \leq \dim U$. For the other direction, assume that $\dim V - \dim W \leq \dim U$. Pick a basis $(u_1, \ldots, u_m)$ for $U$ and extend it to a basis $(u_1, \ldots, u_m, v_1, \ldots, v_k)$ for $V$. Note that $k = \dim V - \dim U \leq \dim W$, by our assumption. Therefore it is possible to pick an independent list $(w_1, \ldots, w_k)$ of length $k$ in $W$. Define a map $T: V \to W$ by setting $T u_i = 0$ for $i = 1, \ldots, m$, and $T v_i = w_i$, for $i = 1, \ldots, k$. Then this map has nullspace equal to $U$. Certainly $U$ is contained in the null space, since each $u_i$ goes to zero. But there can’t be anything else in the nullspace either, for if some linear combination $a_1 v_1 + \cdots + a_k v_k$ goes to zero under $T$, then by linearity we would have $a_1 w_1 + \cdots + a_k w_k = 0$, which forces all $a_i$ to be zero by independence of the $w_i$. Thus the null space of this map $T$ is exactly $U$.

14. First assume that $T$ is injective. We must produce a “left-inverse” to $T$. We pick a basis $(w_1, \ldots, w_m)$ for $\text{Range} T$, extend it to a basis $(w_1, \ldots, w_m, \ldots, w_n)$ and define $S: W \to V$ as follows. Since each of $w_1, \ldots, w_m$ is in the range of $T$, there exists, for each $i$, a $v_i \in V$ with $Tv_i = w_i$ (i = 1, \ldots, m). Now we define our map $S$ by setting $Sw_1 = v_1, \ldots, Sw_m = v_m$, and $Sw_{m+1} = \cdots = Sw_n = 0$. Now we show that $ST$ is the identity map on $V$. For this it is sufficient to show that $\text{Null} ST = 0$. But $\text{Null} ST \subseteq \text{Null} T = 0$ since $T$ is injective. For the other direction, assume there is a map $S$ with $ST$ the identity map on $V$. Suppose $v \in \text{Null} T$. Then $Tv = 0$, so $STv = 0$. But $STv = v$, so $v$ was zero to begin with. This means the null space of $T$ is 0, so $T$ is injective.

15. First assume $T$ is surjective. We produce a “right-inverse” to $T$. Pick a basis $v_1, \ldots, v_n$ for $V$. Then $Tv_1, \ldots, Tv_n$ span the range of $T$, which is all of $W$ by assumption of surjectivity. So we may reduce the list $Tv_1, \ldots, Tv_n$ to a basis for $W$. After possibly reordering the $Tv_i$s we may assume this basis is $Tv_1, \ldots, Tv_k$. Now we define our map $S: W \to V$ using this basis, by setting $S(Tv_i) = v_i$ for $i = 1, \ldots, k$. Then for each basis vector $Tv_i$, we have $(TS)(Tv_i) = T(STv_i) = Tv_i$ so $TS$ is the identity on this basis, hence $TS$ is the identity map on $W$. For the other direction, assume there is such a map $S$. Pick any $w \in W$. Then $w = (TS)w = T(Sw)$, so each $w$ is in the range of $T$, hence $T$ is surjective.

16. First observe that $\text{Null} T \subseteq \text{Null} ST$, since if $Tu = 0$, then also $STu = 0$. By theorem 2.13, we can find a subspace $Y$ of $\text{Null} ST$ such that $\text{Null} ST = \text{Null} T \oplus Y$ (this $Y$ is then also a subspace of $U$). Pick a basis $(u_1, \ldots, u_k)$ for $Y$. Then we have $\dim \text{Null} ST = \dim \text{Null} T + k$. Now, the $Tu_i$ are independent, since if $a_1 Tu_1 + \cdots + a_k Tu_k = 0$, then $a_1 u_1 + \cdots + a_k u_k \in \text{Null} T$, but $\text{Null} T \cap Y = 0$, so this is impossible (this says informally that $T$ is injective when applied only to $Y$). Moreover,
the $Tv_i$ are in $\text{Null } S$ (since $u_i \in \text{Null } ST$), hence can be extended to a basis of $\text{Null } T$. Thus
\[
\text{dim Null } ST = \text{dim Null } T + k \leq \text{dim Null } T + \text{dim Null } S
\]

20. We produce an inverse function $S$. Given an $n \times 1$ column vector, which typesetting requires me to write as a row, call it $(a_1, \ldots, a_n)$, we define $S(a_1, \ldots, a_n)$ to be the vector $a_1 \cdot \cdot \cdot a_n v_n \in V$. Let us check that $ST$ is the identity map on $V$ (by exercise 23, this also shows that $TS$ is the identity, and hence that $S$ and $T$ are indeed inverses). Pick any $v$ in $V$ and write it as $v = c_1 v_1 + \cdots + c_n v_n$. Then $Tv$ is the “column vector” $(c_1, \ldots, c_n)$, and by our definition above, applying $S$ to this gives us $c_1 v_1 + \cdots + c_n v_n$, so $ST$ is the identity map.

22. First assume that both $S$ and $T$ are invertible, with inverse maps $S^{-1}$ and $T^{-1}$, respectively. Then $T^{-1} S^{-1}$ is the inverse to $ST$, since $(ST) (T^{-1} S^{-1}) = S IS^{-1} = I$, and similarly $(T^{-1} S^{-1}) (ST) = I$. Conversely, suppose that $ST$ is invertible. Then it is both surjective and injective. Since it’s injective, $\text{Null } ST = 0$. But $\text{Null } T \subseteq \text{Null } ST = 0$, so $T$ is injective also. By theorem 3.21, this means $T$ is invertible. Similarly, since $ST$ is surjective, $\text{Range } ST = V$. But $\text{Range } S \supseteq \text{Range } ST = V$, so $S$ is surjective, hence invertible. Thus both $T$ and $S$ are invertible.

23. Assume that $ST = I$. Since $I$ is invertible, $ST$ is invertible, so both $T$ and $S$ are surjective and injective, by the previous problem. To check that $TS = I$, we pick any $v \in V$ and show that $TSv = v$. But by surjectivity of $T$, $v = Tu$ for some $u \in V$, so $TSv = TSTu = T I u = Tu = v$, which is what we wanted. The other direction is the same - just swap $S$ and $T$.

24. If $T = c I$, then for any $S$, and any $v \in V$, $STv = S(cv) = cSv$, while $TSv = c I (Sv) = cSv$, so since $v$ was arbitrary, $ST = TS$. The other direction is the hard part. So pick a map $T$, which has the property that $ST = TS$ for every map $S \in \mathcal{L}(V)$. We’re going to apply this assumption to a few special maps. First choose a basis $(v_1, \ldots, v_n)$ for $V$, and define maps $\phi_{ij}: V \rightarrow V$ by
\[
\phi_{ij}(v_k) = \begin{cases} v_j & \text{if } k = i \\ 0 & \text{if } k \neq i \end{cases}
\]
So, for example, the map $\phi_{23}$ sends $v_2$ to $v_3$ and kills all the other basis vectors. This is the abstract/linear map version of the “elementary matrices” $E_{ij}$, which you might have seen before. Now by our assumption on $T$, it commutes with all these $\phi$s, i.e., $T \phi_{ij} = \phi_{ij} T$ for all $i, j$. Now, we want to know what $T$ does to each $v_i$, so pick one of them. We’ll compute $\phi_{ij}Tv_i$ and $T \phi_{ij}v_i$ and set them equal to one another, by our assumption. Firstly, write $Tv_i = a_1 v_1 + \cdots + a_n v_n$. Then apply $\phi_{ij}$: $\phi_{ij}Tv_i = \phi_{ij}(a_1 v_1 + \cdots + a_n v_n) = a_i v_j$, because $\phi_{ij}$ kills all the $v$s except $v_i$, which it sends to $v_j$. On the other hand, we compute $T \phi_{ij}v_i = Tv_j$. Setting them equal to one another shows that $Tv_j = a_i v_j$. This is true for each $j$, so we’ve found that $T$ just scales each basis vector (in terms of matrices, this would mean the matrix for $T$ in this basis is diagonal). But now thinking of $j$ as fixed, and varying $i$, we see that the equation $Tv_j = a_i v_j$ forces all the $a_i$s to be the same (because the left hand side doesn’t involve $i$ at all!). So $a_1 = \cdots = a_n = c$, for some scalar $c$. Thus we’ve found that $Tv_j = cv_j$ for each $v_j$. So $T$ is just scaling by $c$ on the basis vectors. By linearity, $T$ is just scaling by $c$ on all vectors, so $T = c I$.

25. The subset of noninvertible operators in $\mathcal{L}(V)$ is not closed under addition. For instance, take the maps $\phi_{ii}$ of the previous problem (i.e., just those $\phi_{ij}$ where $i = j$). Certainly each
$\phi_{ii}$ is not invertible (it kills all the other $v_j$, so it has $n-1$-dimensional null space). However, $\phi_{11} + \phi_{22} + \cdots + \phi_{nn}$ is the identity map, which is no longer in the set of noninvertible maps.

26. Notice that the first system of equations can be written as $Ax = 0$, where $A$ is the $n \times n$ matrix whose $i,j$ entry is the coefficient $a_{ij}$ in the system of equations, and $x = (x_1, \ldots, x_n) \in \mathbb{F}^n$, and 0 means the zero vector in $\mathbb{F}^n$. Similarly, the system of equations in (b) can be written as $Ax = c$, where $c = (c_1, \ldots, c_n) \in \mathbb{F}^n$. Multiplication by $A$ is a linear map $\mathbb{F}^n \rightarrow \mathbb{F}^n$, so (a) is equivalent to saying that the linear map is injective. On the other hand, the condition in (b) (for $Ax = c$ to have a solution for every $c$) is equivalent to saying that multiplication by $A$ is surjective. But this linear map is an operator on $\mathbb{F}^n$, so by 3.21 its injectivity and surjectivity are equivalent.

Additional Problem: Determine exactly which $2 \times 2$ real matrices give rise to invertible maps $\mathbb{R}^2 \rightarrow \mathbb{R}^2$. Solution: Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be our matrix. We want to give conditions on $a, b, c, d$ that ensure invertibility. The condition is that $ad - bc \neq 0$. Let’s prove that... First, it is true in general that a map is invertible if and only if, when applied to a basis of the domain, it yields a basis for the codomain (reason: you can define the inverse map by simply sending the codomain basis back to the original basis). In our case, take the standard basis $e_1, e_2$ for $\mathbb{R}^2$. Then multiplying by $A$ gives two new vectors $\begin{pmatrix} a \\ c \end{pmatrix}$ and $\begin{pmatrix} b \\ d \end{pmatrix}$. So by the discussion above, we will have an isomorphism precisely when these two columns of $A$ are independent. Let’s investigate their independence... They’re independent if and only if the equation

$$\alpha \begin{pmatrix} a \\ c \end{pmatrix} + \beta \begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

has a solution with one or both of $\alpha, \beta$ nonzero; without loss of generality we can consider whether $\beta$ is zero or not (since we may assume neither column is zero). The equation above is equivalent to the system

$$\alpha a + \beta b = 0$$
$$\alpha c + \beta d = 0.$$

Multiply the first equation by $c$ and the second by $a$ and subtract, giving

$$\beta ad - \beta bc = \beta(ad - bc) = 0$$

Thus the system has a nontrivial solution with $\beta$ nonzero if and only if $ad - bc = 0$. Equivalently, the columns of $A$ are independent if and only if $ad - bc \neq 0$. 

3