Chapter 2

3. *Proof.* By the dependence of $(v_1+w,\ v_2+w,\ \cdots,\ v_n+w)$, there is some sequence $a_1, \cdots, a_n$ of real numbers, not all 0, such that

$$a_1(v_1 + w) + \cdots + a_n(v_n + w) = 0.$$

Rearranging terms,

$$a_1v_1 + \cdots + a_nv_n = -(a_1 + \cdots + a_n)w.$$

Since the a_1 are not all 0 and $(v_1, \cdots, v_n)$ is independent, it follows that the LHS of the above equation is not equal to 0. Therefore, on the RHS, $a_1 + \cdots + a_n$ is also non-0 (and, incidentally, w is non-0). So we may divide across:

$$frac{-a_1}{a_1} + \cdots + a_nv_1 + \cdots + \frac{-a_n}{a_1 + \cdots + a_n}v_n = w.$$

So by definition, $w \in \text{span}(v_1, \cdots, v_n)$, as desired. □

5. *Proof.* Let e_n denote the infinite sequence of elements of $\mathbf{F}$ with all 0s except for a 1 in the n^{th} place. For every n, the sequence $(e_1, e_2, \cdots, e_n)$ is linearly independent: for any $a_1, \cdot, a_n \in \mathbf{F}$ not all equal to 0,

$$a_1e_1 + \cdots + a_ne_n = (a_1, a_2, \cdots, a_{n-1}, a_n, 0, 0, \cdots) \neq (0, 0, \cdots).$$

We conclude from problem 7, below, that $\mathbf{F}^\infty$ is infinite dimensional over $\mathbf{F}$. □

7. *Proof.* $\Rightarrow$: Suppose that V is infinite dimensional. We will prove by induction that there exists some sequence $v_1, v_2 \cdots \in V$ such that for every n, the first n of these are independent.

Base case. Because V is infinite dimensional, $V \neq \{0\}$, since $\{0\}$ has dimension 0 over any field. Therefore, there is some non-zero $v_1 \in V$, and so (v_1) is independent.

Inductive step. Assume that $(v_1, \cdots, v_n)$ is an independent set of vectors in V. By our premise, these vectors cannot span V, otherwise V would have dimension at most n; so there is some $v_{n+1} \in V - \text{span}(v_1, \cdots, v_n)$. In particular, this means that $v_{n+1} \neq 0$. We will show that $(v_1, \cdots, v_n, v_{n+1})$ is independent.

Consider any $a_1, \cdots, a_{n+1}$ and suppose that

$$a_1v_1 + \cdots + a_nv_n + a_{n+1}v_{n+1} = 0.$$

Rearranging terms,

$$a_1v_1 + \cdots + a_nv_n = -a_{n+1}v_{n+1}.$$

If a_{n+1} were non-0 then we could divide across by it, and we would have written v_{n+1} as a linear combination of the v_i with $i \leq n$. By our definition of v_{n+1} as not belonging to the span of the other vectors, this is not possible. So $a_{n+1} = 0$. Thus,

$$a_1v_1 + \cdots + a_nv_n = 0,$$

and by our inductive hypothesis that $(v_1, \cdots, v_n)$ is independent, it follows that all of the a_i equal 0. We conclude that $(v_1, \cdots, v_{n+1})$ is independent, as desired.

By the principle of mathematical induction (PMI), there exists a sequence $v_1, v_2, \cdots$ such that for every n, the first n of these are independent, as desired.

$\Leftarrow$: Now, suppose that there exists a sequence $v_1, v_2, \cdots \in V$ such that for every n, the first n of these are independent, and we will show that V is infinite dimensional. By a theorem in Axler, each spanning set for a vector space as at least as large as any linearly independent set. Since V contains a linearly independent set of size n for every positive integer n, it can have no finite spanning set. So by definition, the space is infinite dimensional. □

8. Every vector in U is of the form

$$(3x_2, x_2, 7x_4, x_4, x_5) = x_2(3,1,0,0,0) + x_4(0,0,7,1,0) + x_5(0,0,0,0,1).$$

Moreover, distinct values of $x_2,\ x_4$, and x_5 always result in distinct combinations. Therefore the set $\{(3,1,0,0,0);\ (0,0,7,1,0);\ (0,0,0,0,1)\}$ is a basis for U.

9. This is true.

Proof. Let $p_0 = 1;\ p_1 = x;\ p_2 = x^2 + x^3;\ p_3 = x^3$. This is a basis for $\mathcal{P}_4(\mathbf{F})$. □

10. *Proof.* First, we will not address the problem in the case $n = 0$. In this case, the claim is either trivial or nonsense, depending on our whether we define the empty direct sum. So we assume $n \geq 1$.

By a theorem in Axler, V has some basis $B = (b_1, \cdots, b_n)$. Let $U_i = \operatorname{span}(b_i)$ for each i from 1 to n. Now we will show that the U_i are direct-summable. By a theorem in Axler, it suffices to show that a sum $u_1 + \cdots + u_n$ of one vector from each of the spaces U_i comes out to 0 only if all of the chosen vectors u_i are 0. If $u_i \in U_i$ for each i then each $u_i = a_i b_i$ for some a_i. Thus, if

$$u_1 + \cdots + u_n = 0$$

then

$$a_1 b_1 + \cdots + a_n b_n = 0.$$

By the independence of B, this means that all of the a_i equal 0, and thus all of the u_i equal 0. Therefore, the direct sum $U_1 \oplus \cdots \oplus U_n$ is defined. Since this direct sum equals a subspace of V containing the basis B, it must equal V itself. □

11. *Proof.* U has some basis $B = (b_1, \cdots, b_n)$. Since $\dim(U) = \dim(V) = n$, it follows that B is an independent set in V of size $\dim(V)$. Therefore, by a proposition in Axler, B is a basis for V. Since $U = \operatorname{span}(B) = V$, we conclude that $U = V$, as desired. □

13. *Proof.* By a (major!) theorem in Axler $\dim(U) + \dim(V) - \dim(U \cap V) = \dim(U + V)$. Plugging everything in, this gives $\dim(U \cap V) = 0$. The only 0-dimensional vector space is the trivial space $\{0\}$. Thus, $U \cap V = \{0\}$. □

14. *Proof.* By the same formula as in the previous problem,

$$\dim(U) + \dim(W) - \dim(U \cap W) = 10 - \dim(U \cap W) = \dim(U + W) \leq 9.$$

Therefore $\dim(U \cap W) \geq 1$, so in particular, $U \cap W$ is non-trivial. □

15. This formula is not true in general.

Proof by counterexample. We consider three subspaces of $\mathbf{R}^3$. Let $U_1 = \operatorname{span}((1,0,0),(0,1,0)$; $U_2 = \operatorname{span}((1,0,0),(0,0,1))$; and $U_3 = \operatorname{span}((1,0,0),(0,1,1))$. Then for $i \neq j$, the intersection $U_i \cap U_j = \operatorname{span}((1,0,0))$. Furthermore, $U_1 \cap U_2 \cap U_3 = \operatorname{span}((1,0,0))$. Thus,

$$\dim(U_1)+\dim(U_2)+\dim(U_3)-\dim(U_1\cap U_2)-\dim(U_1\cap U_3)-\dim(U_2\cap U_3)+\dim(U_1\cap U_2\cap U_3)=6$$
$$\neq \dim(U_1+U_2+U_3)=3.$$

□

16. *Proof by induction on m. Base case.* In the $m=1$ case, this formula reduces to $\dim(U_1) \leq \dim(U_1)$, which is trivial.

Inductive step. We assume that

$$\dim(U_1+\cdots+U_m) \leq \dim(U_1)+\cdots+\dim(U_m)$$

and we will prove that

$$\dim(U_1+\cdots+U_m+U_{m+1}) \leq \dim(U_1)+\cdots+\dim(U_m)+\dim(U_{m+1}).$$

Let $W = U_1+\cdots+U_m$. By a theorem in Axler and our inductive hypothesis,

$$\begin{aligned}\dim(W+U_{m+1}) &= \dim(W)+\dim(U_{m+1})-\dim(W\cap U_{m+1})\\ &\leq \dim(W)+\dim(U_{m+1})\\ &\leq (\dim(U_1)+\cdots+\dim(U_m))+\dim(U_{m+1}),\end{aligned}$$

as desired.

Therefore, by the PMI, the inequality holds for every $m \geq 1$. □

Extra problem

Proof. Consider the space $U = \operatorname{span}(B_1 \cup B_2)$. This space U is a subspace of V, and because B_1 and B_2 span W_1 and W_2 respectively, these two spaces are subsets of U. As Axler observes, $W_1 \oplus W_2$ is the smallest subspace of V that contains both W_1 and W_2. Thus, $W_1 \oplus W_2 \subseteq U$. But by our premise, $W_1 \oplus W_2 = V$. Thus $V \subseteq U$, and finally, $U = V$.

Let

$$d_1 = |B_1| = \dim(W_1) \text{ and } d_2 = |B_2| = \dim(W_2).$$

Then $|B_1 \cup B_2| \leq d_1+d_2$. By a theorem in Axler, $\dim(V) = d_1+d_2$. Therefore, since $B_1 \cup B_2$ spans V, it is at least as big as a basis for V; in particular, $|B_1 \cup B_2| \geq d_1+d_2$. It follows that $|B_1+B_2| = d_1+d_2$ exactly. Since $B_1 \cup B_2$ spans V and has cardinality equal to $\dim(V)$, we conclude that $B_1 \cup B_2$ is a basis for V. □