Solutions to Homework #1.

Chapter 1.

1. \[
\frac{1}{a+ib} = \frac{(a-ib)}{(a+ib)(a-ib)} = \frac{a-ib}{a^2+b^2} = \frac{a}{a^2+b^2} + i\frac{-b}{a^2+b^2}
\]

3. Recall that \(-v = (-1)v\). Thus \(-(-v) = ((-1)(-1)v) = (1)v = v\).

4. Suppose \(av = 0\) but \(a \neq 0\). Then \(v = (a/a)v = (1/a)(av) = (1/a)0 = 0\).

6. Take \(U = \{(m,n) \mid m, n \in \mathbb{Z}\} \subset \mathbb{R}^2\). It is nonempty and closed under addition and taking additive inverses, but it is not a subspace since it is not closed under scalar multiplication by \(1/2\).

7. Take \(U = \{(x,0) \mid x \in \mathbb{R}\} \cup \{(0,y) \mid y \in \mathbb{R}\} \subset \mathbb{R}^2\). It is nonempty and closed under scalar multiplication, but it is not a subspace since it is not closed under addition: \((1,1) = (1,0) + (0,1) \notin U\) even though \((1,0), (0,1) \in U\).

8. Let \(U_i \subset V\), for \(i \in I\), be a collection of subspaces. To see \(\bigcap_{i \in I} U_i \subset V\) is a subspace we check:
   
   1. \(\bigcap_{i \in I} U_i\) is closed under addition: if \(u, v \in U_i\), then \(u, v \in U_i\), for all \(i \in I\). Thus \(u+v \in U_i\), for all \(i \in I\), and so \(u+v \in \bigcap_{i \in I} U_i\).
   
   2. \(\bigcap_{i \in I} U_i\) is closed under scalar multiplication: if \(v \in U_i\), then \(v \in U_i\), for all \(i \in I\). Thus for any \(a \in F\), we have \(av \in U_i\), for all \(i \in I\), and so \(av \in \bigcap_{i \in I} U_i\).
   
   3. \(\bigcap_{i \in I} U_i\) contains the additive identity \(0\): we have \(0 \in U_i\), for all \(i \in I\), and so \(0 \in \bigcap_{i \in I} U_i\).

9. Let \(U_1, U_2 \subset V\) be subspaces.

   Suppose \(U_1 \subset U_2\). Then \(U_1 \cup U_2 = U_2\) and so \(U_1 \cup U_2\) is a subspace.

   Suppose \(U_2 \subset U_1\). Then \(U_1 \cup U_2 = U_1\) and so \(U_1 \cup U_2\) is a subspace.

   Conversely, suppose \(U_1 \not\subset U_2\) and \(U_2 \not\subset U_1\). Thus there exist vectors \(u_1 \in U_1, u_1 \not\in U_2\) and \(u_2 \in U_2, u_2 \not\in U_1\). Now let us prove that \(U_1 \cup U_2\) is not a subspace. We will show that \(w = u_1 + u_2 \notin U_1 \cup U_2\) even though \(u_1, u_2 \in U_1 \cup U_2\). Let us prove this by contradiction: so suppose \(w = u_1 + u_2 \in U_1 \cup U_2\). Then we have \(w = u_1 + u_2 \in U_1\) or \(w = u_1 + u_2 \notin U_2\). In the first case, we have \(u_2 = w - u_1 \in U_1\) since \(w, u_1 \in U_1\); but \(u_2 \not\in U_1\), a contradiction. In the second case, we have \(u_1 = w - u_2 \in U_2\) since \(w, u_2 \in U_2\); but \(u_1 \not\in U_2\), a contradiction.

13. Here is a counterexample disproving the statement. Take \(V = \mathbb{R}^2, U_1 = \{(x,0)\}, U_2 = \{(0,y)\}\), and \(W = \{(t,t)\}\). Then \(U_1 + W = V = U_2 + W\) but \(U_1 \neq U_2\).

14. Take \(W = \{q(z) = c_0 + c_1 z + \cdots + c_m z^m \mid c_2 = c_5 = 0\}\). Then clearly any polynomial \(p(z) = a_0 + a_1 z + \cdots + a_m z^m\) can be written uniquely as a sum

\[
p(z) = q(z) + (a_2 z^2 + a_5 z^5)
\]

where we set \(q(z) = p(z) - a_2 z^2 + a_5 z^5\).

15. Here is a counterexample disproving the statement. Take \(V = \mathbb{R}^2, U_1 = \{(x,0)\}, U_2 = \{(0,y)\}\), and \(W = \{(t,t)\}\). Then \(U_1 \oplus W = V = U_2 \oplus W\) but \(U_1 \neq U_2\).
Additional problem. Find all subspaces of \( \mathbb{R}^2 \).

Let \( W \subset \mathbb{R}^2 \) be a subspace. We will show that \( W \) is the zero subspace \( \{0\} \), a line through the origin \( \{av \mid v \neq 0 \in \mathbb{R}^2\} \), or the whole vector space \( \mathbb{R}^2 \).

If \( W \) contains only 0, then \( W = \{0\} \) and we are done.

Else \( W \) contains some vector \( v \neq 0 \). Thus \( W \) contains the line \( \{av \mid v \neq 0 \in \mathbb{R}^2\} \).

If \( W = \{av \mid v \neq 0 \in \mathbb{R}^2\} \), then we are done.

Else \( W \) contains some vector \( w \neq av \). We will show that in this case we have \( W = \mathbb{R}^2 \). Take any vector \( u \in \mathbb{R}^2 \). We will show \( u \in W \) by finding \( c, d \in \mathbb{R} \) such that \( u = cv + dw \). Write \( u = (u_1, u_2) \), \( v = (v_1, v_2) \), and \( w = (w_1, w_2) \). Then we seek to solve the system

\[
\begin{bmatrix}
  u_1 \\
  u_2 \\
\end{bmatrix} = \begin{bmatrix}
  v_1 & w_1 \\
  v_2 & w_2 \\
\end{bmatrix} \begin{bmatrix}
  c \\
  d \\
\end{bmatrix}
\]

Since \( v \neq 0 \) and \( w \neq av \), we can solve the system by

\[
\begin{bmatrix}
  c \\
  d \\
\end{bmatrix} = \frac{1}{v_1 w_2 - w_1 v_2} \begin{bmatrix}
  w_2 & -w_1 \\
  -v_2 & v_1 \\
\end{bmatrix} \begin{bmatrix}
  u_1 \\
  u_2 \\
\end{bmatrix}
\]