Solutions to Homework #1.

Chapter 1.

1.
$$\frac{1}{(a+ib)} = \frac{(a-ib)}{(a+ib)(a-ib)} = \frac{a-ib}{a^2+b^2} = \frac{a}{a^2+b} + i\frac{-b}{a^2+b^2}$$

3. Recall that $-v = (-1)v$. Thus $-(-v) = (-1)((-1)v) = ((-1)(-1))v = (1)v = v$.

4. Suppose $av = 0$ but $a \neq 0$. Then $v = (a/a)v = (1/a)(av) = (1/a)0 = 0$.

6. Take $U = \{(m,n) \,|\, m,n \in \mathbb{Z}\} \subset \mathbb{R}^2$. It is nonempty and closed under addition and taking additive inverses, but it is not a subspace since it is not closed under scalar multuplication by $1/2$.

7. Take $U = \{(x,0) \,|\, x \in \mathbb{R}\} \cup \{(0,y) \,|\, y \in \mathbb{R}\} \subset \mathbb{R}^2$. It is nonempty and closed under scalar multiplication, but it is not a subspace since it is not closed under addition: $(1,1) = (1,0)+(0,1) \notin U$ even though $(1,0),(0,1) \in U$.

8. Let $U_i \subset V$, for $i \in I$, be a collection of subspaces. To see $\cap_{i\in I} U_i \subset V$ is a subspace we check:

(1) $\cap_{i\in I} U_i$ is closed under addition: if $u, v \in \cap_{i\in I} U_i$, then $u, v \in U_i$, for all $i \in I$. Thus $u+v \in U_i$, for all $i \in I$, and so $u+v \in \cap_{i\in I} U_i$.

(2) $\cap_{i\in I} U_i$ is closed under scalar multiplication: if $v \in \cap_{i\in I} U_i$, then $v \in U_i$, for all $i \in I$. Thus for any $a \in F$, we have $av \in U_i$, for all $i \in I$, and so $av \in \cap_{i\in I} U_i$.

(3) $\cap_{i\in I} U_i$ contains the additive identity 0: we have $0 \in U_i$, for all $i \in I$, and so $0 \in \cap_{i\in I} U_i$.

9. Let $U_1, U_2 \subset V$ be subspaces.

Suppose $U_1 \subset U_2$. Then $U_1 \cup U_2 = U_2$ and so $U_1 \cup U_2$ is a subspace.

Suppose $U_2 \subset U_1$. Then $U_1 \cup U_2 = U_1$ and so $U_1 \cup U_2$ is a subspace.

Conversely, suppose $U_1 \not\subset U_2$ and $U_2 \not\subset U_1$. Thus there exist vectors $u_1 \in U_1, u_1 \notin U_2$ and $u_2 \in U_2, u_2 \notin U_1$. Now let us prove that $U_1 \cup U_2$ is not a subspace. We will show that $w = u_1 + u_2 \notin U_1 \cup U_2$ even though $u_1, u_2 \in U_1 \cup U_2$. Let us prove this by contradiction: so suppose $w = u_1 + u_2 \in U_1 \cup U_2$. Then we have $w = u_1 + u_2 \in U_1$ or $w = u_1 + u_2 \notin U_2$. In the first case, we have $u_2 = w - u_1 \in U_1$ since $w, u_1 \in U_1$; but $u_2 \notin U_1$, a contradiction. In the second case, we have $u_1 = w - u_2 \in U_2$ since $w, u_2 \in U_2$; but $u_1 \notin U_2$, a contradiction.

13. Here is a counterexample disproving the statement. Take $V = \mathbb{R}^2$, $U_1 = \{(x,0)\}$, $U_2 = \{(0,y)\}$, and $W = \{(t,t)\}$. Then $U_1 + W = V = U_2 + W$ but $U_1 \neq U_2$.

14. Take $W = \{q(z) = c_0 + c_1 z + \cdots + c_m z^m \,|\, c_2 = c_5 = 0\}$. Then clearly any polynomial $p(z) = a_0 + a_1 z + \cdots + a_m z^m$ can be written uniquely as a sum

$$p(z) = q(z) + (a_2 z^2 + a_5 z^5)$$

where we set $q(z) = p(z) - a_2 z^2 + a_5 z^5$.

15. Here is a counterexample disproving the statement. Take $V = \mathbb{R}^2$, $U_1 = \{(x,0)\}$, $U_2 = \{(0,y)\}$, and $W = \{(t,t)\}$. Then $U_1 \oplus W = V = U_2 \oplus W$ but $U_1 \neq U_2$.

Additional problem. Find all subspaces of $\mathbb{R}^2$.

Let $W \subset \mathbb{R}^2$ be a subspace. We will show that W is the zero subspace $\{0\}$, a line through the origin $\{av \mid v \neq 0 \in \mathbb{R}^2\}$, or the whole vector space $\mathbb{R}^2$.

If W contains only 0, then $W = \{0\}$ and we are done.

Else W contains some vector $v \neq 0$. Thus W contains the line $\{av \mid v \neq 0 \in \mathbb{R}^2\}$.

If $W = \{av \mid v \neq 0 \in \mathbb{R}^2\}$, then we are done.

Else W contains some vector $w \neq av$. We will show that in this case we have $W = \mathbb{R}^2$. Take any vector $u \in \mathbb{R}^2$. We will show $u \in W$ by finding $c, d \in \mathbb{R}$ such that $u = cv + dw$. Write $u = (u_1, u_2)$, $v = (v_1, v_2)$, and $w = (w_1, w_2)$. Then we seek to solve the system

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix}$$

Since $v \neq 0$ and $w \neq av$, we can solve the system by

$$\begin{bmatrix} c \\ d \end{bmatrix} = \frac{1}{v_1 w_2 - w_1 v_2} \begin{bmatrix} w_2 & -w_1 \\ -v_2 & v_1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$