Homework 13 Solutions

14. Define $T \in \mathcal{L}\left(\mathbb{C}^4\right)$ by $T(e_i) = 7e_i$ for $i = 1, 2$ and $T(e_i) = 8e_i$ for $i = 3, 4$, and extend by linearity.

15. Since 5 and 6 are the only eigenvalues of T and because V is a vector space over $\mathbb{C}$, it follows that the characteristic polynomial for T has the form

$$p_T(x) = (x-5)^a(x-6)^b$$

for some $a, b \in \mathbb{N}$. Furthermore, it follows that $a, b \geq 1$ since 5 and 6 must both have algebraic multiplicity at least 1, and that $a + b = n$ since p_T has degree n. Thus, $a, b \leq n-1$, and we can conclude that p_T divides $(x-5)^a(x-6)^b$. Since $p_T(T) = 0$ by the Cayley Hamilton theorem, it follows that $(T-5I)^{n-1}(T-6I)^{n-1} = 0$, since $(x-5I)^{n-1}(x-6I)^{n-1}$ is also divisible by p_T. $\square$.

16. Claim: V has a basis consisting of eigenvectors of T if and only if every generalized eigenvector of T is an eigenvector of T.

proof: ($\Longleftarrow$). Assume that every generalized eigenvector of T is also an eigenvector of T. Then by Theorem 8.25, there exists a basis β of V consisting of generalized eigenvectors of T. Since by assumption every generalized eigenvector of T is actually an eigenvector of T, this basis β is actually a basis of eigenvectors of T.

($\Longrightarrow$). Conversely, for the other direction, Suppose V has a basis $\beta = \{v_1, \ldots, v_n\}$ consisting of eigenvectors of T. That is $Tv_i = \lambda_i v_i$ for some λ_i not necessarily distinct. Then if $w \in V$ is a generalized eigenvector of T corresponding to the eigenvalue λ, there exists constants $c_1, \ldots, c_n \in \mathbb{C}$ such that $w = c_1v_1 + \cdots c_nv_n$. Furthermore, since w is a generalized eigenvector of T, $(T-\lambda)^n w = 0$. Hence,

$$0 = (T-\lambda I)^n w = (\lambda_1 - \lambda)^n c_1 v_1 + \cdots + (\lambda_n - \lambda)^n c_n v_n.$$

which implies that $(\lambda_i - \lambda)^n c_i = 0$ for all i. So if $c_i \neq 0$, then it must be the case that $\lambda_i = \lambda$. It follows then that w is a linear combination of vectors in $E_\lambda(T)$, and thus an eigenvector of T $\square$.

17. This follows directly from an application of theorem 8.26 followed by an application of theorem 6.27/

21. Let $T \in \mathcal{L}(\mathbb{C}^3)$. be defined by $T(e_1) = 0$, $T(e_2) = e_1$, $T(e_3) = 0$ and extend by linearity. Then $T^2 = 0$, but $T \neq 0$, so it follows that the min polynomial of T is z^2. $\square$

22. Let $T \in \mathcal{L}(\mathbb{C}^4)$ be given by $T(e_1) = e_1$, $T(e_2) = e_1 + e_2$ and $T(e_3) = T(e_4) = 0$ and extend by linearity. Then

$$[T] = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since 1 and 0 are the eigenvalues of T, we know the min polynomial of T must be of the form $m_T(z) = z^a(z-1)^b$ for $a, b \geq 1$. One can check that $T(T-I)^2 = 0$, but $T(T-1) \neq 0$ since $T(T-I)e_2 = e_1$. so it follows that $m_T(z) = z(z-1)^2$.

23. Suppose V is a vector space over $\mathbb{C}$ and let $T \in \mathcal{L}(V)$. Claim: V has a basis consisting of eigenvalues of T if and only if the min polynomial of of T has no repeated roots. Proof: ($\Longrightarrow$) Suppose $\lambda_1, \ldots \lambda_k$ are the distinct eigenvalues of T and that V has a basis consisting of eigenvectors of T. We will show that the min polynomial of T, $m_T(x) = (x-\lambda_1)\ldots(x-\lambda_k)$. Now let $\{v_1, \ldots, v_n\}$ be a basis of V consisting of eigenvectors of T. Then if $i \in \{1, \ldots, n\}$, $Tv_i = \lambda_l v_i$ for some $l \in \{1, \ldots k\}$. Since $(T - \lambda_i I)$ and

$(T-\lambda_j I$ commute for any i and j, so it follows that $(T-\lambda_1 I)\dots(T-\lambda_k I)v_i = 0$. Thus $(T-\lambda_1 I)\dots(T-\lambda_k I) = 0$ since it is equal to 0 on a basis of V. This implies that $m_T(x)|(x-\lambda_1)\dots(x-\lambda_k)$. Furthermore, m_T must divide $(x-\lambda_1)\dots(x-\lambda_k)$, because the eigenvalues of T are roots of m_T. Since two monic polynomials that divide each other must be equal, it follows that $m_T(x) = (x-\lambda_1)\dots(x-\lambda_k)$ as desired. $(\Longleftarrow)$ Conversely, assume $m_T(x) = (x-\lambda_1)\dots(x-\lambda_k)$, where again $\lambda_1, \dots \lambda_k$ are the distinct eigenvalues of T. We know that V is a direct sum of the generalized eigenspaces: $G_{\lambda_i} := G_{\lambda_i}(T)$. So we are done if we can show that each generalized eigenspace of T is actually equal to the corresponding eigenspace. This is equivalent to showing that $(T-\lambda_i I)|_{G_{\lambda_i}} = T|_{G_{\lambda_i}} - \lambda_i I = 0$ i.e. that $G_{\lambda_i} \subseteq null(T-\lambda_i I)$. Now, we know that $m_T(T) = (T-\lambda_1 I)\dots(T-\lambda_k I) = 0$, and thus $(T|_{G_{\lambda_i}} - \lambda_1 I)\dots(T|_{G_{\lambda_i}} - \lambda_k I) = 0$ for each i. Because G_{λ_i} is T-invariant, and λ_j is not an eigenvalue of $T|_{G_{\lambda_i}}$ when $j \neq i$, it follows that $(T|_{G_{\lambda_i}} - \lambda_j I)$ is invertible as an operator on G_{λ_i} for $j \neq i$. If we multiply both sides of the equation by $(T|_{G_{\lambda_i}} - \lambda_j I)^{-1}$, for each $j \neq i$, this implies that $(T|_{G_{\lambda_i}} - \lambda_i I) = 0$ as desired. $\square$

24. Suppose T is normal, and the min polynomial of T is given by $m_T(z) = (z-\lambda)^k p(z)$ where $p(\lambda) \neq 0$. That is λ is repeated as a root k times in m_T. We will show that $(T-\lambda I)p(T) = 0$, which is a monic polynomial that zeros T and divides m_T, and hence must equal T. This will show that $k = 1$. To do this, not that

$$0 = m_T(T) = (T-\lambda I)^k p(T)$$

which shows that range $p(T) \subseteq null(T-\lambda I)^k$. Now $T - \lambda I$ is normal because T is and in exercise 7 of chapter 7 we showed that $null(T-\lambda I)^k = null(T-\lambda I)$, so range $p(T) \subseteq null(T-\lambda I)$, and hence $(T-\lambda I)p(T) = 0$, which is what we wanted to show. $\square$.

25. Suppose $p(T)$ is the monic polynomial of smallest degree such that $p(T)v = 0$ for some $v \in V$. Let m_T denote the min polynomial of T. By the division algorithm, there exists polynomials d and r with $\deg r < \deg p$ such that

$$m_T = pd + r$$

since $m_T = 0$, $m_T(v) = 0$, so $0 = m_T(v) = p(T)d(T)v + r(T)v = 0 + r(T)v$. so $r(T)v = 0$, which implies that $r(T) = 0$, otherwise this would be a polynomial of degree less than $\deg p$ that sends v to 0, which would contradict our assumption. Hence $m_T = pd$ which shows that P divides m_T. $\square$.

26. A useful fact when finding an example for this problem is that the degree of $(x-\lambda)$ in the min polynomial of T is the size of the largest Jordan block corresponding to the eigenvalue λ. Hence we want the Jordan canonical form J of T to be of the form

$$J = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

That is J has Jordan blocks of size 1 for the eigenvalues 0 and 3, and a Jordan block of size 2 for the eigenvalue 1. A linear transformation that has this Jordan canonical form is given by $T(e_1) = e_1$, $T(e_2) = e_1 + e_2$, $T(e_3) = 3e_3$ and $T(e_4) = 0$. Then $[T] = J$. Then m_T must be divisible by $x(x-3)(x-1)$ since 0,1 and 3 are all eigenvalues of T. It is easy to check that $T(T-I)(T-3I) \neq 0$, however $T(T-I)^2(T-3I) = 0$, which shows that $m_T(z) = z(z-3)(z-1)^2$.

27. T will have minpolynomial $m_T(z) = z(z-1)(z-3)$ if and only if the eigenvalues of T are $0, 1, 3$, and the Jordan blocks of T all have size 1 (see the comment in the solution

to problem 26.). In other words, T must be diagonalizable. In order for $p_T(z) = z(z-1)^2(z-3)$, 1 must have algebraic multiplicity 2, and hence geometric multiplicity 2 since T is diagonalizable. One can check that T meets the above conditions where $T(e_1) = e_1$, $T(e_2) = e_2$, $T(e_3) = 3e_3$ and $T(e_4) = 0$. This is easy to verify because $[T]$ is given by the diagonal matrix with $1, 1, 3, 0$ down the main diagonal.

28. Choose $T \in \mathcal{L}(\mathbb{C}^n)$ so that $[T]$ equals the matrix shown in the problem. That is $T(e_i) = e_{i+1}$ for $i = 1, \ldots, n-1$ and $T(e_n) = -a_0e_1 - a_1e_2 - \cdots - a_{n-1}e_n$. Hence, $T^i(e_1) = e_{i+1}$, for $i = 1, \ldots, n-1$, and $T^n(e_1) = Te_n$. Thus $\{e_1, Te_1, \ldots, T^{n-1}e_1\}$ is linearly independent, so in particular, $p(T)e_1 \neq 0$ for all monic polynomials p of degree less than n. So m_T has degree n, hence $m_T = p_T$. Furthermore, from the equations above, $T^n(e_1) = -a_{n-1}T^{n-1}e_1 - \cdots - a_1Te_1 - a_0$. The a_i given are unique by the independence of the vectors $T^i(e_1)$ for $i = 1, \ldots, n-1$, so $p(z) = a_0 + a_1z + \cdots + a_nz^n$ is the only monic degree n polynomial that sends e_1 to 0. since p_T is a monic polynomial of degree n that sends e_1 to 0, it follows that $p_T = p$. $\square$.