Homework 13 Solutions

14. Define $T \in \mathcal{L}(\mathbb{C}^4)$ by $T(e_i) = 7e_i$ for $i = 1, 2$ and $T(e_i) = 8e_i$ for $i = 3, 4$, and extend by linearity.

15. Since 5 and 6 are the only eigenvalues of $T$ and because $V$ is a vector space over $\mathbb{C}$, it follows that the characteristic polynomial for $T$ has the form

$$p_T(x) = (x - 5)^a(x - 6)^b$$

for some $a, b \in \mathbb{N}$. Furthermore, it follows that $a, b \geq 1$ since 5 and 6 must both have algebraic multiplicity at least 1, and that $a + b = n$ since $p_T$ has degree $n$. Thus, $a, b \leq n - 1$, and we can conclude that $p_T$ divides $(x - 5)^a(x - 6)^b$. Since $p_T(T) = 0$ by the Cayley Hamilton theorem, it follows that $(T - 5I)^{n-a}(T - 6I)^{n-b} = 0$, since $(x - 5I)^{n-1}(x - 6I)^{n-1}$ is also divisible by $p_T$. $\square$

16. Claim: $V$ has a basis consisting of eigenvectors of $T$ if and only if every generalized eigenvector of $T$ is an eigenvector of $T$.

proof: $(\Longleftrightarrow)$. Assume that every generalized eigenvector of $T$ is also an eigenvector of $T$. Then by Theorem 8.25, there exists a basis $\beta$ of $V$ consisting of generalized eigenvectors of $T$. Since by assumption every generalized eigenvector of $T$ is actually an eigenvector of $T$, this basis $\beta$ is actually a basis of eigenvectors of $T$.

$(\Rightarrow)$. Conversely, for the other direction, Suppose $V$ has a basis $\beta = \{v_1, \ldots, v_n\}$ consisting of eigenvectors of $T$. That is $Tv_i = \lambda_i v_i$ for some $\lambda_i$ not necessarily distinct. Then if $w \in V$ is a generalized eigenvector of $T$ corresponding to the eigenvalue $\lambda_i$, there exists constants $c_1, \ldots, c_n \in \mathbb{C}$ such that $w = c_1 v_1 + \cdots + c_n v_n$. Furthermore, since $w$ is a generalized eigenvector of $T$, $(T - \lambda)^n w = 0$. Hence,

$$0 = (T - \lambda I)^n w = (\lambda_1 - \lambda)^n c_1 v_1 + \cdots + (\lambda_n - \lambda)^n c_n v_n,$$

which implies that $(\lambda_i - \lambda)^n c_i = 0$ for all $i$. So if $c_i \neq 0$, then it must be the case that $\lambda_i = \lambda$. It follows then that $w$ is a linear combination of vectors in $E_\lambda(T)$, and thus an eigenvector of $T$. $\square$

17. This follows directly from an application of theorem 8.26 followed by an application of theorem 6.27.

21. Let $T \in \mathcal{L}(\mathbb{C}^3)$ be defined by $T(e_1) = 0$, $T(e_2) = e_1$, $T(e_3) = 0$ and extend by linearity. Then $T^2 = 0$, but $T \neq 0$, so it follows that the min polynomial of $T$ is $z^2$. $\square$

22. Let $T \in \mathcal{L}(\mathbb{C}^4)$ be defined by $T(e_1) = e_1$, $T(e_2) = e_1 + e_2$ and $T(e_3) = T(e_4) = 0$ and extend by linearity. Then

$$[T] = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since 1 and 0 are the eigenvalues of $T$, we know the min polynomial of $T$ must be of the form $m_T(z) = z^a(z - 1)^b$ for $a, b \geq 1$. One can check that $T(T - I)^2 = 0$, but $T(T - I) \neq 0$ since $T(T - I)e_2 = e_1$. So it follows that $m_T(z) = z(z - 1)^2$.

23. Suppose $V$ is a vector space over $\mathbb{C}$ and let $T \in \mathcal{L}(V)$. Claim: $V$ has a basis consisting of eigenvectors of $T$ if and only if the min polynomial of $T$ has no repeated roots.

Proof: $(\Rightarrow)$ Suppose $\lambda_1, \ldots, \lambda_k$ are the distinct eigenvalues of $T$ and that $V$ has a basis consisting of eigenvectors of $T$. We will show that the min polynomial of $T$, $m_T(x) = (x - \lambda_1)\cdots(x - \lambda_k)$. Now let $\{v_1, \ldots, v_n\}$ be a basis of $V$ consisting of eigenvectors of $T$. Then if $i \in \{1, \ldots, n\}$, $Tv_i = \lambda_i v_i$ for some $l \in \{1, \ldots, k\}$. Since $(T - \lambda_i I)$ and
(T − λI) commute for any i and j, so it follows that (T − λ1I)...(T − λkI)vi = 0, Thus (T − λ1I)...(T − λkI) = 0 since it is equal to 0 on a basis of V. This implies that mT(x)||(x − λ1)...(x − λk). Furthermore, mT must divide (x − λ1)...(x − λk), because the eigenvalues of T are roots of mT. Since two monic polynomials that divide each other must be equal, it follows that mT(x) = (x − λ1)...(x − λk) as desired. (⇐) Conversely, assume mT(x) = (x − λ1)...(x − λk), where again λ1, ..., λk are the distinct eigenvalues of T. We know that V is a direct sum of the generalized eigenspaces: \(G_{λi} := G_{λi}(T)\). So we are done if we can show that each generalized eigenspace of T is actually equal to the corresponding eigenspace. This is equivalent to showing that \((T − λI)|_{G_{λi}} = T|_{G_{λi}} − λI) = 0\) i.e. that \(G_{λi} \subseteq \text{null}(T − λiI)\). Now, we know that \(mT(T) = (T − λ1I)...(T − λkI) = 0\), and thus \((T|_{G_{λi}} − λI)|...|_{G_{λi}} − λI) = 0\) for each i. Because \(G_{λi}\) is T-invariant, and \(λi\) is not an eigenvalue of \(T|_{G_{λi}}\) when \(j \neq i\), it follows that \((T|_{G_{λi}} − λiI)\) is invertible as an operator on \(G_{λi}\) for \(j \neq i\). If we multiply both sides of the equation by \((T|_{G_{λi}} − λiI)^{-1}\), for each \(j \neq i\), this implies that \((T|_{G_{λi}} − λiI) = 0\) as desired. \(\square\)

24. Suppose T is normal, and the min polynomial of T is given by \(mT(z) = (z − λ)^kp(z)\) where \(p(λ) \neq 0\). That is \(λ\) is repeated as a root \(k\) times in \(mT\). We will show that \((T − λI)p(T) = 0\), which is a monic polynomial that zeros T and divides \(mT\), and hence must equal T. This will show that \(k = 1\). To do this, not that

\[
0 = mT(T) = (T − λI)^kp(T)
\]

which shows that range \(p(T) \subseteq \text{null}(T − λI)^k\). Now \(T − λI\) is normal because T is and in exercise 7 of chapter 7 we showed that \(\text{null}(T − λI)^k = \text{null}(T − λI)\), so range \(p(T) \subseteq \text{null}(T − λI)\), and hence \((T − λI)p(T) = 0\), which is what we wanted to show. \(\square\)

25. Suppose \(p(T)\) is the monic polynomial of smallest degree such that \(p(T)v = 0\) for some \(v \in V\). Let \(mT\) denote the min polynomial of T. By the division algorithm, there exists polynomials \(d\) and \(r\) with \(\text{deg}r < \text{deg}p\) such that

\[
mT = pd + r
\]

since \(mT = 0\), \(mT(v) = 0\), so \(0 = mT(v) = p(T)d(T)v + r(T)v = 0 + r(T)v\), so \(r(T)v = 0\), which implies that \(r(T) = 0\), otherwise this would be a polynomial of degree less than \(\text{deg}p\) that sends \(v\) to 0, which would contradict our assumption. Hence \(mT = pd\) which shows that \(P\) divides \(mT\). \(\square\)

26. A useful fact when finding an example for this problem is that the degree of \((x − λ)\) in the min polynomial of T is the size of the largest Jordan block corresponding to the eigenvalue \(λ\). Hence we want the Jordan canonical form \(J\) of T to be of the form

\[
J = \begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 3 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

That is \(J\) has Jordan blocks of size 1 for the eigenvalues 0 and 3, and a Jordan block of size 2 for the eigenvalue 1. A linear transformation that has this Jordan canonical form is given by \(T(e1) = e1, T(e2) = e1 + e2, T(e3) = 3e3\) and \(T(e4) = 0\). Then \(|T| = J\). Then \(mT\) must be divisible by \(x(x − 3)(x − 1)\) since 0,1 and 3 are all eigenvalues of T. It is easy to check that \(T(T − I)(T − 3I) \neq 0\), however \(T(T − I)^2(T − 3I) = 0\), which shows that \(mT(z) = z(z − 1)(z − 3)\).

27. T will have minpolynomial \(mT(z) = z(z − 1)(z − 3)\) if and only if the eigenvalues of T are 0,1,3, and the Jordan blocks of T all have size 1 (see the comment in the solution
to problem 26.). In other words, $T$ must be diagonalizable. In order for $p_T(z) = z(z-1)^2(z-3)$, 1 must have algebraic multiplicity 2, and hence geometric multiplicity 2 since $T$ is diagonalizable. One can check that $T$ meets the above conditions where $T(e_1) = e_1$, $T(e_2) = e_2$, $T(e_3) = 3e_3$ and $T(e_4) = 0$. This is easy to verify because $[T]$ is given by the diagonal matrix with 1, 1, 3, 0 down the main diagonal.

28. Choose $T \in L(\mathbb{C}^n)$ so that $[T]$ equals the matrix shown in the problem. That is $T(e_i) = e_{i+1}$ for $i = 1, \ldots, n-1$ and $T(e_n) = -a_0 e_1 - a_1 e_2 - \cdots - a_{n-1} e_n$. Hence, $T^i(e_1) = e_{i+1}$, for $i = 1, \ldots, n-1$, and $T^n(e_1) = Te_n$. Thus $\{e_1, Te_1, \ldots, T^{n-1}e_1\}$ is linearly independent, so in particular, $p(T)e_1 \neq 0$ for all monic polynomials $p$ of degree less than $n$. So $m_T$ has degree $n$, hence $m_T = p_T$. Furthermore, from the equations above, $T^n(e_1) = -a_{n-1} T^{n-1} e_1 - \cdots - a_1 Te_1 - a_0$. The $a_i$ given are unique by the independence of the vectors $T^i(e_1)$ for $i = 1, \ldots, n-1$, so $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ is the only monic degree $n$ polynomial that sends $e_1$ to 0. Since $p_T$ is a monic polynomial of degree $n$ that sends $e_1$ to 0, it follows that $p_T = p$. \qed