1. a) Suppose that $T$ were self-adjoint. Then, the Spectral Theorem tells us that there would exist an orthonormal basis of $P_2(\mathbb{R})$, $(p_1, p_2, p_3)$, consisting of eigenvectors of $T$. It is straightforward to see, by inspection of $T$, that the eigenvalues of $T$ are $\lambda = 0, 1$ and that

$$T(v) = 0 \iff v \in \text{null}(T) = \{a_0 + a_2x^2 \mid a_0, a_2 \in \mathbb{R}\},$$

$$T(v) = v \iff v \in \text{span}(x).$$

Thus, we have $p_1, p_2 \in \text{null}(T)$, with $\langle p_1, p_2 \rangle = 0$ and $||p_1|| = ||p_2|| = 1$. The remaining eigenvector $p_3$ must be an element of $\text{span}(x)$, so that $p_3 = cx$, for some (nonzero) $c \in \mathbb{R}$ such that $||p_3|| = 1$. Moreover, we would require that $\text{null}(T) \subset \text{span}(x)^\perp$, since eigenvectors associated to distinct eigenvalues are orthogonal. Therefore, $\text{null}(T) = \text{span}(x)^\perp$, by dimension considerations ($\dim \text{span}(x)^\perp = \dim P_2 - \dim \text{span}(x) = 3 - 1 - 2$).

However, if $p = a_0 + a_2x^2 \in \text{span}(x)^\perp$ then

$$0 = \langle a_0 + a_2x^2, x \rangle = \int_0^1 (a_0 + a_2x^2) x \, dx = \frac{1}{4}(2a_0 + a_2) \implies a_2 = -2a_0$$

so that $p = a_0(1 - 2x^2)$. That is, we have shown that $\text{span}(1 - 2x^2) = \text{span}(x)^\perp \cap \text{null}(T) = \text{null}(T)$, since $\text{null}(T) = \text{span}(x)^\perp$. Then, we would have

$$2 = \dim \text{null}(T) = \dim \text{span}(1 - 2x^2) = 1$$

which is absurd. Hence, our assumption that $T$ is self-adjoint is false.

b) Theorem 6.47 requires that the matrix of $T^*$ (relative to $C \subset W$ and $B \subset V$) is the conjugate transpose of the matrix of $T$ (relative to $B \subset V$ and $C \subset W$) if both $B$ and $C$ are orthonormal. However, the basis $B = C = (1, x, x^2)$ of $P_2$ is not orthonormal (with respect to the given inner product) so that we are not contradicting Theorem 6.47. If we chose an orthonormal basis of $P_2$, call it $A$ (obtained by Gram-Schmidt process on $(1, x, x^2)$, for example), then we would find $[T]_A \neq [\overline{T}]_A$.

2. This is false. This would imply that the matrices $A, B$ of two self-adjoint operators $T, S$ (relative to an orthonormal basis) would satisfy

$$(AB)^* = AB,$$

where for a square matrix $C$ we are writing $C^* = C^t$. However, $(AB)^* = B^*A^* = BA$, since $T, S$ are self-adjoint. So, we need to only find two non-commuting self adjoint operators - we can take the following operators on Euclidean space $\mathbb{C}^2$

$$T : \mathbb{C}^2 \rightarrow \mathbb{C}^2 ; \ x \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \ x, \quad S : \mathbb{C}^2 \rightarrow \mathbb{C}^2 ; \ x \mapsto \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \ x$$

You can check that $TS(e_1) \neq ST(e_1)$, where $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

3. a) Let $T, S \in L(V)$, be self-adjoint operators on $V$, a real inner product space. Then, the zero operator on $V$, $Z : V \rightarrow V ; \ v \mapsto 0_v$ is self-adjoint; we have $(T + S)^* = T^* + S^* = T + S$, so that $T + S$ is self-adjoint; if $c \in \mathbb{R}$ then $(cT)^* = cT^* = cT$, so that $cT$ is self-adjoint.

Hence, the set of self-adjoint operators on a real inner product space is a subspace of $L(V)$.  

b) If $T$ is self-adjoint operator on the complex inner product space $V$, then $(\sqrt{-1}T)^* = -\sqrt{-1}T \neq \sqrt{-1}T$. Hence, the set of self-adjoint operators is not closed under scalar multiplication.

4. Let $P \in L(V)$ be such that $P^2 = P$.

($\Rightarrow$) Suppose that $P$ is an orthogonal projection. Then, $\text{range}(P) = \text{null}(P)^\perp$. Moreover, the only eigenvalues of $P$ are $\lambda = 0, 1$ (this was proved in a previous HW exercise) and

$$P(v) = v \iff v \in \text{range}(P)$$

(this is true of any projection) so that $\text{range}(P)$ consists of eigenvectors with eigenvalue $\lambda = 1$. Hence, we can find an orthonormal basis $(u_1, \ldots, u_k)$ of $\text{range}(P)$ (using Gram-Schmidt applied to any basis of $\text{range}(P)$) and an orthonormal basis $(v_1, \ldots, v_l)$ of $\text{null}(P)$ (using Gram-Schmidt applied to any basis of $\text{null}(P)$). Then, $(u_1, \ldots, u_k, v_1, \ldots, v_l)$ is an orthonormal basis of $V$ consisting of eigenvectors of $P$. Hence, if $V$ is a real inner product space then $P$ is self-adjoint, by the Spectral Theorem. If $V$ is complex inner product space then, for any $v \in V$, we can write $v = u + z$, $u \in \text{range}(P), z \in \text{null}(P)$ so that

$$\langle P(v), v \rangle = \langle u, u + z \rangle = \langle u, u \rangle + \langle u, z \rangle = ||u||^2 + 0 \in \mathbb{R}$$

Hence, $P$ is self-adjoint when $V$ is complex inner product space.

($\Rightarrow$) Suppose that $P$ is self-adjoint. Then, we have $\text{null}(P) = \text{null}(P^*)$ and

$$\text{range}(P) = \text{null}(P^*)^\perp = \text{null}(P)^\perp \implies V = \text{null}(P) \oplus \text{null}(P)^\perp = \text{null}(P) \oplus \text{range}(P)$$

Since $P$ is self-adjoint then there is a basis of $V$ consisting of orthonormal vectors of $P$ - call it $(u_1, \ldots, u_k, v_1, \ldots, v_l)$, where $P(u_i) = 0_V$ and $P(v_i) \neq 0_V$. Hence, $\dim \text{null}(P) = k$ (i.e. we are saying that the $u$s are an o.n. basis of $\text{null}(P)$) and, since $\text{span}(v_1, \ldots, v_l) \subset \text{null}(P)^\perp = \text{range}(P)$ and $\dim \text{range}(P) = \dim V - \dim \text{null}(P) = (k + l) - k = l$, we see that $\text{span}(v_1, \ldots, v_l) = \text{range}(P)$. Thus, $(v_1, \ldots, v_l)$ is an orthonormal basis of $\text{range}(P)$ consisting of eigenvectors of $P$ with nonzero associated eigenvalues. As we are assuming that $P^2 = P$, we must have that the only eigenvalues of $P$ are $\lambda = 0, 1$, so that the only nonzero eigenvalue is $\lambda = 1$. Hence, for every $u \in \text{range}(P)$ we have $P(u) = u$. Thus, since we can write $v = z + u$, with $z \in \text{null}(P), u \in \text{range}(P)$, we see that

$$P(v) = P(z + u) = P(z) + P(u) = 0_V + u,$$

so that $P$ is a projection onto $\text{range}(P)$ with $\text{null}(P) = \text{range}(P)^\perp$ - hence, it is an orthogonal projection.

5. Let $V$ be an inner product space, take $(v_1, \ldots, v_n)$ an orthonormal basis of $V$ (so that $n \geq 2$). Consider the normal operators $T, S \in L(V)$ defined as follows:

$$T(v_1) = 3v_1, \quad T(v_2) = v_2, \quad T(v_i) = 0_V, \quad i \geq 3,$$

$$S(v_1) = v_2, \quad S(v_2) = -v_1, \quad S(v_i) = 0_V, \quad i \geq 3.$$ 

Then, the $(n \times n)$ matrices of $T, S$ with respect to the given basis are

$$A = \begin{bmatrix} 3 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$
Since, $A = \overline{A}^t$, we have $T = T^*, \text{ and as } \overline{B}B^t = \overline{B}^tB$, we have $SS^* = S^*S$, giving that both $T$ and $S$ are normal.

Now, we see that the matrix of $T + S$ is

$$A + B = \begin{bmatrix} 3 & -1 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

and this last matrix is not diagonalisable (so that $T + S$ is not diagonalisable: indeed, the eigenvalues of $T + S$ are $\lambda = 0, 2$ and

$$\text{null}(T + S) = \text{span}(v_3, \ldots, v_n)$$

while the $\lambda = 2$ eigenspace is $\text{span}(v_1 + v_2)$. If $T + S$ were to be diagonalisable then we would need to have two linearly independent eigenvectors with eigenvalue $\lambda = 2$, which obviously can’t be the case. Hence, $T + S$ is not normal (it isn’t diagonalisable).

6. Let $T \in L(V)$ be normal. Then, we must have that $\text{null}(T) = \text{null}(T^*)$ (this is at the top of p.131). Hence, $\text{range}(T) = \text{null}(T^*)^\perp = \text{null}(T)^\perp = \text{range}(T^*)$.

7. There are a couple of ways to proceed:

Proof I) Let $B$ be an orthonormal basis of $V$ consisting of eigenvectors of $T$ (it exists by the Spectral Theorem). Then, we have the matrix of $T$ relative to $B$ is

$$[T]_B = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}$$

where $\lambda_1, \ldots, \lambda_n$ are eigenvalues of $T$ (counted with multiplicity). Let’s suppose that $\lambda_1 = \ldots = \lambda_k = 0$, and $\lambda_i \neq 0$, for $i > k$. Thus, $\text{dim null}(T) = k$. Now, for any $j \geq 1$ we have

$$[T^j]_B = [T]_B^j = [T]_B = \begin{bmatrix} \lambda_{1}^{j} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_{n}^{j} \end{bmatrix}$$

and $\lambda_{i}^{j} = 0 \implies \lambda_{r} = 0 \implies r \in \{1, \ldots, k\}$. Hence, $\text{null}(T^j) = \text{span}(v_1, \ldots, v_k) = \text{null}(T)$, for each $j \geq 1$.

Now, since, for each $j \geq 1$,

$$\text{dim range}(T) = \text{dim} V - \text{dim null}(T) = \text{dim} V - \text{dim null}(T^j) = \text{dim range}(T^j)$$

and $\text{range}(T^j) \subset \text{range}(T)$, we see that $\text{range}(T) = \text{range}(T^j)$ follows from $\text{null}(T) = \text{null}(T^j)$.

Proof II) As $T$ is normal then we have $\text{null}(T) = \text{null}(T^*)$ (see p.131). Hence, we have

$$\text{range}(T) = \text{null}(T^*)^\perp = \text{null}(T)^\perp \implies V = \text{null}(T) \oplus \text{range}(T)$$
In particular, \( \text{null}(T) \cap \text{range}(T) = \{0\} \). Let's prove \( \text{null}(T^j) = \text{null}(T) \), for every \( j \geq 1 \), by induction: the case \( j = 1 \) is trivial. Assume the result hold for \( j = s \) - we'll show it holds for \( j = s + 1 \). Since \( \text{null}(T) \subset \text{null}(T^{s+1}) \) always holds, we need only show that \( \text{null}(T) \supset \text{null}(T^{s+1}) \). So, let \( z \in \text{null}(T^{s+1}) \). Then,

\[
0 = T^{s+1}(z) = T(T^s(z)) \implies T^s(z) \in \text{null}(T) \cap \text{range}(T) = \{0\}
\]

\[
\implies z \in \text{null}(T^s) = \text{null}(T), \text{ by induction.}
\]

Hence, \( \text{null}(T^{s+1}) \subset \text{null}(T) \) and the result is proved.

8. The requirements on \( T \) imply that the vectors \((1, 2, 3)\) and \((2, 5, 7)\) are eigenvectors of \( T \). However, with respect to the dot product on \( \mathbb{R}^3 \), we see that

\[
(1, 2, 3) \cdot (2, 5, 7) = 2 + 10 + 21 = 33 \neq 0
\]

so that eigenvectors corresponding to distinct eigenvalues are not orthogonal, contradicting Corollary 7.8.

9. \((\Rightarrow)\) Suppose that \( T \) is self-adjoint. Then, by Proposition 7.1 we see that all eigenvalues of \( T \) are real.

\((\Leftarrow)\) Suppose that all eigenvalues of the normal operator \( T \) are real. Then, by the (complex) Spectral Theorem, we can find an orthonormal basis \( B \) of \( V \) consisting of eigenvectors of \( T \). Hence, we have the matrix of \( T \) relative to \( B \) is

\[
[T]_B = \begin{bmatrix}
\lambda_1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \lambda_n
\end{bmatrix}, \text{ where } \lambda_1, \ldots, \lambda_n \in \mathbb{R}.
\]

Then, we have that

\[
[T^*]_B = [T^t]_B = [T]_B \implies T = T^*.
\]

Hence, \( T \) is self-adjoint.

10. Since \( T \) is normal, there is an orthonormal basis \( B \) of \( V \) consisting of eigenvectors of \( T \). Then, we have

\[
[T]_B = \begin{bmatrix}
\lambda_1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \lambda_n
\end{bmatrix}, \quad \text{and} \quad [T^i]_B = \begin{bmatrix}
\lambda_1^i & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \lambda_n^i
\end{bmatrix}.
\]

Hence, if \( T^8 = T^9 \) then we must have

\[
[T^8]_B = \begin{bmatrix}
\lambda_1^8 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \lambda_n^8
\end{bmatrix} = \begin{bmatrix}
\lambda_1^9 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \lambda_n^9
\end{bmatrix} = [T^9]_B
\]

so that, for each \( i = 1, \ldots, n \)

\[
\lambda_i^8 = \lambda_i^9 \implies \lambda_i^8(1 - \lambda_i) = 0
\]
In particular, each eigenvalue $\lambda_i$ is either equal to 1 or 0. Since the eigenvalues of $T$ are real then $T$ is self-adjoint (by previous exercise). Moreover, if we assume that $\lambda_1 = \cdots = \lambda_k = 0$ and $\lambda_{k+1} = \cdots = \lambda_n = 1$ then we have

$$[T]_B = \begin{bmatrix} \lambda_1^2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n^2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} = [T]_B$$

so that $T^2 = T$.

11. Let $T$ be normal and $B = (v_1, \ldots, v_n)$ be an orthonormal basis of $V$ consisting of eigenvectors of $T$. Suppose that $[T]_B = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}$.

Then, by the Fundamental Theorem of Algebra, we can find a (complex) square root of $\lambda_i$, for each $i = 1, \ldots, n$. Suppose that $\mu_i^2 = \lambda_i$, for each $i$. Then, define the operators $S \in L(V)$ as follows:

$$S(v_1) = \mu_1 v_1, \ldots, S(v_n) = \mu_n v_n$$

Then, we have

$$[S^2]_B = \begin{bmatrix} \mu_1^2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mu_n^2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} = [T]_B \implies S^2 = T.$$