Solutions to Homework #10.

6. 21, 22, 26, 27, 28, 29, 30, 31, 32

21. The question is asking for the orthogonal projection of the vector $(1, 2, 3, 4)$ onto the subspace U. To do this we first need an orthonormal basis for U. Applying Gram-Schmidt to the given spanning list for U produces the orthonormal basis $\frac{1}{\sqrt{2}}(1, 1, 0, 0), \frac{1}{\sqrt{5}}(0, 0, 1, 2)$ for U. Then the orthogonal projection of $(1, 2, 3, 4)$ onto U can be computed by

$$\left\langle (1,2,3,4), \frac{1}{\sqrt{2}}(1,1,0,0)\right\rangle \frac{1}{\sqrt{2}}(1,1,0,0) + \left\langle (1,2,3,4), \frac{1}{\sqrt{5}}(0,0,1,2)\right\rangle \frac{1}{\sqrt{5}}(0,0,1,2) = (\frac{3}{2}, \frac{3}{2}, \frac{11}{5}, \frac{22}{5})$$

22. Let U be the subspace of $P_{\leq 3}(\mathbb{R})$ consisting of polynomials p satisfying $p(0) = p'(0) = 0$. Then the problem is asking for the orthogonal projection of the vector $2 + 3x$ onto the subspace U. We first apply the Gram-Schmidt procedure to the basis (x^2, x^3) for U, which gives the orthonormal basis $(\sqrt{5}x^2, 6\sqrt{7}x^3 - 5\sqrt{7}x^2)$.

Now we look at our vector $2 + 3x$, and first take its inner products with the two vectors above

$$\langle 2+3x, \sqrt{5}x^2 = \int_0^1 (2+3x)(\sqrt{5}x^2)dx = \frac{17\sqrt{5}}{12}$$
$$\langle 2+3x, 6\sqrt{7}x^3 - 5\sqrt{7}x^2\rangle = \int_0^1 (2+3x)(6\sqrt{7}x^3 - 5\sqrt{7}x^2)dx = -\frac{29\sqrt{7}}{60}$$

These will be the coefficients when we project this vector onto U, namely:

$$(\frac{17\sqrt{5}}{12})\sqrt{5}x^2 + (-\frac{29\sqrt{7}}{60})(6\sqrt{7}x^3 - 5\sqrt{7}x^2) = \frac{1}{10}x^2(240 - 203x)$$

26. Whatever the adjoint T^*a is, it must satisfy $\langle Tu, a\rangle = \langle u, T^*a\rangle$, which can be rewritten as $\langle u, v\rangle a = \langle u, T^*a\rangle$. So setting $T^*a = \overline{a}v$ works, and by uniqueness of adjoints, this is the formula for T^*.

27. Since the matrix for T in terms of the standard basis is

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}$$

and since the standard basis is an orthonormal basis (with respect to the standard inner product on $\mathbb{F}^n$), we can find the adjoint by just taking the conjugate transpose of this matrix. So the matrix of T^* in the standard basis is

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

So the formula for T^* is $T^*(z_1, \ldots, z_n) = (z_2, z_3, \ldots, z_n, 0)$.

28. First note that, for any λ, $(T - \lambda I)^* = T^* - \overline{\lambda} I$, by properties of adjoints. Now suppose that λ is an eigenvalue of T, so that Null $(T - \lambda I)$ is not zero. But then $\left(\text{Range } (T - \lambda I)^*\right)^\perp$ is nonzero, since these are equal by 6.46(c). Thus Range $(T - \lambda I)^*$ is not equal to all of V, and hence $(T - \lambda I)$ is neither injective nor surjective. By the comment at the beginning, this means $T^* - \overline{\lambda} I$ is not injective, so $\overline{\lambda}$ is an eigenvalue of T^*. The argument in the other direction is the same, just replacing T by T^* and λ by $\overline{\lambda}$

29. First assume that U is invariant under T, and pick $w \in U^\perp$. We must show that $T^* w$ is in $U^\perp$ also. To do this, pick any $u \in U$. By the definition of adjoint, we have $\langle u, T^* w \rangle = \langle Tu, w \rangle$, and this is zero since $Tu \in U$ and $w \in U^\perp$. The proof of the other direction is identical: just replace U by $U^\perp$ and T by T^*, and use the facts that $(U^\perp)^\perp = U$ and $T^{**} = T$.

30. For (a), T is injective iff Null $T = 0$ iff (Range $T^*)^\perp = 0$ (by 6.46) iff Range $T^* = W$ iff T^* is surjective. For (b), T is surjective iff Range $T = W$ iff (Null $T^*)^\perp = W$ (by 6.46) iff Null $T^* = 0$ iff T^* is injective.

31. The rank-nullity theorem applied to T, and to T^* gives us the following two equations:

$$\dim V = \dim \text{Null } T + \dim \text{Range } T \tag{1}$$
$$\dim W = \dim \text{Null } T^* + \dim \text{Range } T^* \tag{2}$$

We can rewrite the first one as

$$\dim(\text{Null } T)^\perp = \dim \text{Range } T,$$

and since (Null $T)^\perp$ = Range T^*, this gives the second equality. To prove the other equality, we replace $\dim \text{Range } T^*$ by $\dim V - \dim(\text{Null } T)^\perp$ in the second equation above, and reorganize.

32. Left-multiplication by A gives a linear map L_A from $\mathbb{R}^n$ to $\mathbb{R}^m$. Moreover, the matrix of L_A with respect to the standard (orthonormal) basis is just A. So by theorem 6.47, the adjoint map L_A^* from $\mathbb{R}^m$ to $\mathbb{R}^n$ is just left multiplication by A^T. By the previous problem, the ranges of L_A and L_A^* have the same dimension. But the range of L_A is just the span of the columns of A, as you can see by applying L_A to each of the standard basis vectors. Similarly, the range of L_A^* is the span of the columns of A^T, which is the same as the span of the rows of A (except that they're row vectors rather than column vectors). Therefore the row span and the column span of A have the same dimension. (PS - the point of this problem is to convince you that the abstract point of view is useful! Look up "row rank=column rank" in a standard linear algebra text to see a more traditional proof - it's much messier. But of course, we had to develop a lot of abstract theory first...)