Name (Last, First): ____________________

Student ID: ____________________

Circle your GSI and section:

Scerbo	8am	200 Wheeler	Forman	2pm	3109 Etcheverry
Scerbo	9am	3109 Etcheverry	Forman	4pm	3105 Etcheverry
McIvor	12pm	3107 Etcheverry	Melvin	5pm	24 Wheeler
McIvor	11am	3102 Etcheverry	Melvin	4pm	151 Barrows
Mannisto	12pm	3 Evans	Mannisto	11am	3113 Etcheverry
Wayman	1pm	179 Stanley	McIvor	2pm	179 Stanley
Wayman	2pm	81 Evans			

If none of the above, please explain: ____________________

This exam consists of 10 problems, each worth 10 points, of which you must complete 8. **Choose two problems not to be graded by crossing them out in the box below.** You must justify every one of your answers unless otherwise directed.

Problem	Maximum Score	Your Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total Possible	80	

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1. Let V be a nonzero finite-dimensional complex inner product space. Suppose $T : V \to V$ is a linear transformation with adjoint $T^* : V \to V$.

Decide if the following assertions are ALWAYS TRUE or SOMETIMES FALSE. You need not justify your answer.

i. There exists an eigenvalue of T.

ii. There exists an orthonormal basis of V such that the matrix of T^* is upper-triangular.

iii. There exists an orthonormal basis of V such that the matrix of TT^* is diagonal.

iv. If $W \subset V$ is T-invariant, then $W^\perp \subset V$ is T^*-invariant.

v. If T is self-adjoint, then T is an orthogonal projection.

vi. If T is an orthogonal projection, then T^* is self-adjoint.

vii. If λ is an eigenvalue of T, then $\bar{\lambda}$ is an eigenvalue of T^*.

viii. If v is an eigenvector of T, then v is an eigenvector of T^*.

ix. If T is nilpotent, and $null(T)$ is T^*-invariant, then $T = 0$.

x. If TT^* is nilpotent, then $T = 0$.

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2. Let V be a vector space over a field F.

(a) State what it means for a list of vectors $v_1, \ldots, v_k$ to be linearly independent. State the definition of the span of a list of vectors $v_1, \ldots, v_k$.

(b) Let $v_1, \ldots, v_k$ be a linearly independent list of vectors and let v be a vector contained in the span of $v_1, \ldots, v_k$. Prove that the list of vectors $v_1, \ldots, v_k, v$ is not linearly independent.

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3. Let V be a finite-dimensional complex inner product space. Suppose $T : V \to V$ is a normal operator such that each of its eigenvalues satisfies $|\lambda| \leq 1$. Prove that $\|Tv\| \leq \|v\|$, for any $v \in V$.

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4. Consider $\mathbb{C}^3$ with the standard Euclidean inner product. Determine whether each of the following operators $T : \mathbb{C}^3 \to \mathbb{C}^3$ is a projection ($T^2 = T$), orthogonal projection ($T^2 = T$ and $null(T) \perp range(T)$), or neither. You need not justify your answer.

a. T is normal with eigenvalues $0, 1$.

b. $T(v) = \langle v, w \rangle w$, with $w = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}) \in \mathbb{C}^3$.

c. T has eigenvectors $(i, 0, 0), (i, -i, 0), (i, -i, i)$ with respective eigenvalues $1, 1, 0$.

d. T has eigenvectors $(i, 0, -1), (1, -i, 0), (1, i, -i)$ with respective eigenvalues $0, 0, 1$.

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5. Find the minimal polynomial, characteristic polynomial, and Jordan form of the linear transformation $T : \mathbb{C}^5 \to \mathbb{C}^5$ given by the matrix

$$T = \begin{pmatrix} 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 \end{pmatrix}$$

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6. Consider the following matrices:

$$T_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad T_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad T_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$T_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad T_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad T_6 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Which of the matrices has minimal polynomial $m(z) = z^4 - z^2$? Give a brief justification for those matrices which do not.

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7. Find all possible Jordan forms of an operator $T : \mathbb{C}^8 \to \mathbb{C}^8$ given the following information:

$$\dim null((T-1)^2) = 4 \qquad \dim range(T-1) = 6 \qquad \dim null((T-2)^3) = 4$$

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8. Find a basis that puts the operator given by the matrix

$$T = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

into Jordan form. What is the Jordan form?

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9. Consider $\mathbb{C}^3$ with the standard Euclidean inner product.

a. Find an orthonormal basis of

$$U = \{(x, y, z) \in \mathbb{C}^3 \,|\, x + y + z = 0\}$$

b. Find the vector in U closest to the vector $(1, 1, 0)$.

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10. Let V, W be finite-dimensional complex spaces. Prove or give a counterexample to the following assertion:

If $T : V \to W$ is a linear transformation with $\dim range(T) = 1$, then we can find a vector $w \in W$ and a linear functional $f : V \to \mathbb{C}$ such that

$$T(v) = f(v)w, \qquad \text{for any } v \in V$$