Name (Last, First):  
Student ID:  

Circle your GSI and section:

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<td>8am</td>
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<td>McIvor</td>
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If none of the above, please explain:  

This exam consists of 10 problems, each worth 10 points, of which you must complete 8.  
**Choose two problems not to be graded by crossing them out in the box below.**  
You must justify every one of your answers unless otherwise directed.

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1. Let $V$ be a nonzero finite-dimensional complex inner product space. Suppose $T : V \to V$ is a linear transformation with adjoint $T^* : V \to V$.

Decide if the following assertions are ALWAYS TRUE or SOMETIMES FALSE. You need not justify your answer.

i. There exists an eigenvalue of $T$.

ii. There exists an orthonormal basis of $V$ such that the matrix of $T^*$ is upper-triangular.

iii. There exists an orthonormal basis of $V$ such that the matrix of $TT^*$ is diagonal.

iv. If $W \subset V$ is $T$-invariant, then $W^\perp \subset V$ is $T^*$-invariant.

v. If $T$ is self-adjoint, then $T$ is an orthogonal projection.

vi. If $T$ is an orthogonal projection, then $T^*$ is self-adjoint.

vii. If $\lambda$ is an eigenvalue of $T$, then $\bar{\lambda}$ is an eigenvalue of $T^*$.

viii. If $v$ is an eigenvector of $T$, then $v$ is an eigenvector of $T^*$.

ix. If $T$ is nilpotent, and $null(T)$ is $T^*$-invariant, then $T = 0$.

x. If $TT^*$ is nilpotent, then $T = 0$. 
2. Let $V$ be a vector space over a field $F$.

(a) State what it means for a list of vectors $v_1, \ldots, v_k$ to be linearly independent. State the definition of the span of a list of vectors $v_1, \ldots, v_k$.

(b) Let $v_1, \ldots, v_k$ be a linearly independent list of vectors and let $v$ be a vector contained in the span of $v_1, \ldots, v_k$. Prove that the list of vectors $v_1, \ldots, v_k, v$ is not linearly independent.
3. Let $V$ be a finite-dimensional complex inner product space. Suppose $T : V \to V$ is a normal operator such that each of its eigenvalues satisfies $|\lambda| \leq 1$. Prove that $\|Tv\| \leq \|v\|$, for any $v \in V$. 

Name (Last, First): ________________________________________________
4. Consider \( \mathbb{C}^3 \) with the standard Euclidean inner product. Determine whether each of the following operators \( T : \mathbb{C}^3 \rightarrow \mathbb{C}^3 \) is a projection \( (T^2 = T) \), orthogonal projection \( (T^2 = T \) and \( \text{null}(T) \perp \text{range}(T) \)), or neither. You need not justify your answer.

a. \( T \) is normal with eigenvalues 0, 1.

b. \( T(v) = \langle v, w \rangle w \), with \( w = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}) \in \mathbb{C}^3 \).

c. \( T \) has eigenvectors \( (i, 0, 0), (i, -i, 0), (i, -i, i) \) with respective eigenvalues 1, 1, 0.

d. \( T \) has eigenvectors \( (i, 0, -1), (1, -i, 0), (1, i, -i) \) with respective eigenvalues 0, 0, 1.
5. Find the minimal polynomial, characteristic polynomial, and Jordan form of the linear transformation $T : \mathbb{C}^5 \rightarrow \mathbb{C}^5$ given by the matrix

$$ T = \begin{pmatrix} 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 \end{pmatrix} $$
6. Consider the following matrices:

\[
T_1 = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}, \quad
T_2 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad
T_3 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \\
T_4 = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad
T_5 = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix}, \quad
T_6 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix}
\]

Which of the matrices has minimal polynomial \( m(z) = z^4 - z^2 \)? Give a brief justification for those matrices which do not.
7. Find all possible Jordan forms of an operator $T : \mathbb{C}^8 \to \mathbb{C}^8$ given the following information:

$$\dim \text{null}((T - 1)^2) = 4 \quad \dim \text{range}(T - 1) = 6 \quad \dim \text{null}((T - 2)^3) = 4$$
8. Find a basis that puts the operator given by the matrix

\[
T = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}
\]

into Jordan form. What is the Jordan form?
9. Consider $\mathbb{C}^3$ with the standard Euclidean inner product.

a. Find an orthonormal basis of

$$U = \{(x, y, z) \in \mathbb{C}^3 \mid x + y + z = 0\}$$

b. Find the vector in $U$ closest to the vector $(1, 1, 0)$. 
10. Let $V, W$ be finite-dimensional complex spaces. Prove or give a counterexample to the following assertion:

If $T : V \to W$ is a linear transformation with $\dim \text{range}(T) = 1$, then we can find a vector $w \in W$ and a linear functional $f : V \to \mathbb{C}$ such that

$$T(v) = f(v)w, \quad \text{for any } v \in V$$