Solutions to HW9

assigned by prof. Nadler

1 5.1 # 3a

1. The characteristic polynomial is $\det(A-\lambda I) = \det\begin{pmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{pmatrix} = (1-\lambda)(2-\lambda) - 6 = \lambda^2 - 3\lambda - 4$. The roots are $\lambda = -1$ and $\lambda = 4$ and these are precisely the eigenvalues of A.
2. The $\lambda = -1$ eigenspace is $\text{span}\{(-1,1)^t\}$. Indeed, the system $(A-(-1)I)x = 0$ is

$$\left(\begin{array}{cc|c} 2 & 2 & 0 \\ 3 & 3 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

   so we can take x_2 to be a free variable and $x_1 = -x_2$, so the solution space is the set of all vectors of the form $(-x_2, x_2)^t$, i.e., the aforementioned span.

   Similarly, the $\lambda = 4$ eigenspace is $\text{span}\{(2,3)^t\}$. This comes from the system

$$\left(\begin{array}{cc|c} -3 & 2 & 0 \\ 3 & -2 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 \end{array}\right)$$

   so that x_2 is free and $x_1 = (2/3)x_2$. Slap an $x_2 = 3$ in there to get an integer vector $(2,3)^t$ as claimed.
3. These vectors correspond to distinct eigenvalues and so are linearly independent. Hence $\beta = \{(-1,1)^t, (2,3)^t\} :=\} b_1, b_2\}$ is a basis of eigenvectors.
4. The calculations $Ab_1 = -b_1$ and $Ab_2 = 4b_2$ show that $[A]_\beta = \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}$. (Perhaps the book would rather us write $[L_A]_\beta$.) If σ is the standard basis of $\mathbf{F}^2$, then of course $[\text{Id}]^\sigma_\beta [A]_\beta [\text{Id}^\beta_\sigma = [A]_\sigma = A$. So the $Q^{-1} = [\text{Id}]^\sigma_\beta$ which is easy to compute: run through the vectors of β and express them in terms of σ - but since σ is the standard basis this is very easy $[b_1]_\sigma = b_1 = (-1,1)^t$ and $[b_2]_\sigma = b_2 = (2,3)^t$. So $Q^{-1} = \begin{pmatrix} -1 & 2 \\ 1 & 3 \end{pmatrix}$. Using the inverse formula we have $Q = \begin{pmatrix} -3/5 & 2/5 \\ 1/5 & 1/5 \end{pmatrix}$. So the diagonalization is

$$\begin{pmatrix} -1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} -3/5 & 2/5 \\ 1/5 & 1/5 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$$

2 5.1 # 5

$(T-\lambda I)v = 0 \Leftrightarrow Tv - \lambda I v = 0 \Leftrightarrow Tv = \lambda v$. (In conjunction with $v \neq 0$ the equation on the left is "$v \in N(T-\lambda I)$ & $v \neq 0$" and the one on the right is "v is an eigenvector with eigenvalue λ.")

3 5.1 # 8

- (a) A linear operator on a finite dimensional vector space is not invertible if and only if it has a nontrivial nullspace (by the dimension theorem - and note that this is not necessarily true in infinite dimensions). Having a nontrival nullspace amounts to the existence of a nonzero v such that $Tv = 0 = 0 \cdot v$, i.e., a $\lambda = 0$ eigenvector.

  [Alternate proof: 0 is an eigenvalue $\Leftrightarrow$ 0 is a root of the characteristic polynomial $\Leftrightarrow \det([L]_\beta - 0(I)) = \det([L]_\beta) = 0$ $\Leftrightarrow L$ not invertible.]
- (b) If $v \neq 0$ and $Tv = \lambda v$, then λ must not be zero (lest T be singular). Then $v = \lambda T^{-1} v$ and so we can divide by λ to get $\lambda^{-1} v = T^{-1} v$ which shows that λ^{-1} is an eigenvalue of T^{-1}. The result follows by symmetry.

- (c) A matrix A is singular if and only if 0 is not an eigenvalue of L_A. [Proof: A is singular $\Leftrightarrow$ L_A is not invertible $\Leftrightarrow$ 0 not an eigenvalue of L_A (using part (a)).]

  If A is invertible and λ is an eigenvalue of L_A then λ^{-1} is an eigenvalue of $L_A^{-1} = L_{A^{-1}}$. [Proof: If A is invertible then L_A is invertible and by (b) λ is an eigenvalue of L_A if and only if λ^{-1} is an eigenvalue of L_A^{-1}.

4 5.1 # 9

If A is diagonal, then $\det(A-\lambda I) = \det \begin{pmatrix} a_{1,1}-\lambda & a_{1,2} & \cdots & a_{1,n} \\ 0 & a_{2,2}-\lambda & \cdots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{n,n}-\lambda \end{pmatrix} = \prod_{i=1}^{n}(a_{i,i}-\lambda)$. The roots of this polynomial are precisely the values on the diagonal.

[I am implicitly using a theorem here, that if A is upper triangular the determinant is the product of the entries on the diagonal. With expansion by minors one can prove this by induction (e.g. expand along the first column $\det A = a_{1,1}\det(A_{1,1}) + 0\cdot(\cdots)$ and note that $A_{1,1}$ (the $(n-1)\times(n-1)$ minor which omits the first row and column) is still upper triangular). With the cooler definition $\det A = \sum_{\sigma\in S_n}(-1)^\sigma \prod_{i=1}^n a_{i,\sigma(i)}$ note that in order for all of $a_{i,\sigma(i)} \neq 0$ we must have $\sigma(i) \geq i$. But the only permutation which has this property is the identity permutation which has positive signum. So $\det A = \prod a_{i,i}$. I included this just as an example since I claimed this definition gives you many better and more direct proofs.]

5 5.1 # 11

- (a) If $A = Q^{-1}(\lambda I)Q$ then $A = \lambda I Q^{-1}Q = \lambda I$.
- (b) Let $b_1, \cdots, b_n$ be a basis of eigenvectors (which by assumption have the same eigenvalue). Then any vector $v = \sum a_i b_i$ has $Av = \sum a_i A b_i = \sum a_i \lambda b_i = \lambda \sum a_i b_I = \lambda v$. So $Av = \lambda I v$ for all v and hence $A = \lambda I$. [For example, apply this with v ranging over the standard basis.]

  [Alternate proof: $[A]_\beta$ is diagonal and hence upper triangular. Applying exercise 9 shows that all the diagonal entries are the same, so that $[A]_\beta = \lambda I$ and hence $A \sim \lambda I$ and part (a) gives the result.]
- (c) The only eigenvalue of this matrix is 1 by exercise 9, and by part (b) if the matrix were diagonalizable it would be a scalar matrix which it isn't.

6 5.1 # 12

- (a) $\det(Q^{-1}AQ - \lambda I) = \det(Q^{-1}AQ - Q^{-1}\lambda IQ) = \det(Q^{-1}(A-\lambda I)Q) = \det(Q^{-1})\det(A-\lambda I)\det(Q) = \det(A-\lambda I)$.

  [Using implicitly that $\det(Q^{-1}) = (\det(Q))^{-1}$. To prove write $\det(I) = 1 = \det(Q^{-1}Q) = \det(Q^{-1})\det(Q)$.]
- (b) Different matrix representations of the same operator are similar. I.e. $[T]_\beta \sim [T]_\gamma$ for any bases β, γ. Result follows by (a).

7 5.1 # 14

$\det(A^t - \lambda I) = \det((A-\lambda I)^t) = \det(A - \lambda I)$.

[Because $(\lambda I)^t = \lambda I$.]

8 5.1 # 20

$f(t) = \det(A - tI)$ so $f(0) = \det(A - 0\cdot I) = \det(A)$. A is invertible $\Leftrightarrow$ $\det(A) = f(0) = a_0 \neq 0$.