Solutions to Homework #7.

Section 3.3

2. (a) Row reducing we get the matrix $\begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix}$ so by letting $x_2 = t_1$ we have $x_1 + 3t_1 = 0$ which forces $x_1 = -3t_1$. Therefore solutions will look like $t_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix}$. A basis for the solutions is $\left\{ \begin{pmatrix} -3 \\ 1 \end{pmatrix} \right\}$ and the dimension is 1.

(b) Row reducing we get $\begin{pmatrix} 1 & 0 & -1/3 \\ 0 & 1 & -2/3 \end{pmatrix}$. Letting $x_3 = t_1$ we see that $x_1 = 1/3t_1$ and $x_2 = 2/3t_1$. Thus, a basis for the solution set is $\left\{ \begin{pmatrix} 1/3 \\ 2/3 \\ 1 \end{pmatrix} \right\}$ and the dimension is 1.

3. (a) First note that $x_1 = 5, x_2 = 0$ is a particular solution of the system. By exercise 2 (a) the solutions to the system will look like $\begin{pmatrix} 5 \\ 0 \end{pmatrix} + t_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix}$ for any $t_1 \in \mathbb{R}$.

(b) First note that $x_1 = 0, x_2 = -1, x_3 = -2$ is a particular solution. By exercise 2 (b) the solutions to the system will look like $\begin{pmatrix} 0 \\ -1 \\ -2 \end{pmatrix} + t_1 \begin{pmatrix} 1/3 \\ 2/3 \\ 1 \end{pmatrix}$ for any $t_1 \in \mathbb{R}$.

8. (a) The extended matrix is $\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -2 & 3 \\ 1 & 0 & 2 & -2 \end{pmatrix}$ which after row operations becomes $\begin{pmatrix} 1 & 0 & 2 & -2 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. The coefficient matrix and the extended matrix both have rank 2 so the system has solutions.

(b) The same procedure as in part (a) shows that the coefficient matrix and the extended matrix both have rank 2 and the system is consistent.

10. Let A be the coefficient matrix. The extended matrix is $A|b$. Note that since $A|b$ is $m \times (n+1)$ it must satisfy $\text{rank}(A|b) \leq m$. On the other hand, adding one extra column to A cannot decrease the rank so $\text{rank}(A|b) \geq \text{rank}(A) = m$. Therefore, $\text{rank}(A) = \text{rank}(A|b) = m$ and by a theorem from the book the system has solutions.

Section 3.4

3. (a) Row reducing the extended matrix one has $\begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -1 \end{pmatrix}$ so the solution is $\begin{pmatrix} 4 \\ -3 \\ -1 \end{pmatrix}$.

(f) Just like in part (a) we row-reduce the extended matrix to get $\begin{pmatrix} 1 & 0 & 0 & -1 & -3 \\ 0 & 1 & 0 & 2 & 3 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$. This forces $x_3 = 1$ and if we let $x_4 = t_1$ we get $x_1 = -3 + t_1, x_2 = 3 - 3t_1$. The solutions therefore look like $\left\{ \begin{pmatrix} -3 \\ 3 \\ 1 \\ 0 \end{pmatrix} + t_1 \begin{pmatrix} 1 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}$.

8. Row reducing the matrix with columns $u_1, \dots, u_8$ we have $\begin{pmatrix} 1 & -3 & 0 & 4 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -4 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. Looking at the pivot columns we find the basis to be $\{u_1, u_3, u_5, u_7\}$.

12. (a) It is immediate to see that S consists of linearly independent vectors which satisfy our two equations.

(b) First let us find a basis for the solution set. The coefficient matrix is $\begin{pmatrix} 1 & -1 & 0 & 2 & -3 & 1 \\ 2 & -1 & -1 & 3 & -4 & 4 \end{pmatrix}$. The row reduced form is $\begin{pmatrix} 1 & 0 & -1 & 1 & -1 & 3 \\ 0 & 1 & -1 & -1 & 2 & 2 \end{pmatrix}$. This forces $x_1 = t_1 - t_2 + t_3 - 3t_4, x_2 = t_1 + t_2 - 2t_3 - 2t_4$ so a basis for the solutions will be $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ -2 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$. Construct a new matrix with the first columns the vectors from S and the rest of the columns the basis elements we just found. This matrix will be $\begin{pmatrix} 0 & 1 & 1 & -1 & 1 & -3 \\ -1 & 0 & 1 & 1 & -2 & -2 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$. The row reduced form of this matrix is $\begin{pmatrix} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ so by looking at the pivot columns we can find that the extra two vectors we need for the basis are $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ -2 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.