Solutions to Homework #5.

Section 2.4.

2.
(a) Not invertible, as the Lemma on p.101 says that if $T : V \to W$ is invertible and V and W are finite-dimensional, then $\dim V = \dim W$.
(b) Not invertible, as in (a).
(c) Invertible. If $\beta = \{e_1, e_2, e_3\}$ is the standard basis for $\mathbb{R}^3$, then

$$[T]_\beta = (\ [T(e_1)]_\beta \mid T(e_2)_\beta \mid [T(e_3)]_\beta\ ) = \begin{pmatrix} 3 & 0 & -2 \\ 0 & 1 & 0 \\ 3 & 4 & 0 \end{pmatrix}.$$

Theorem 2.18 says that T is invertible if and only if $[T]_\beta$ is invertible, and one can row reduce $[T]_\beta$ to see that it is, indeed, invertible.
(d) Not invertible, as in (a).
(e) Not invertible, as in (a).
(f) Invertible. First we show $N(T) = \{0\}$. Suppose

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in N(T).$$

Then

$$T(A) = \begin{pmatrix} a+b & a \\ c & c+d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Looking at the top right and bottom left entries gives $a = 0$ and $c = 0$, and then the other two entries yield $0 = a + b = b$ and $0 = c + d = d$. Thus $A = 0$, and $N(T) = \{0\}$. By the Dimension Theorem, $\operatorname{rank}(T) = \dim M_{2\times2}(\mathbb{R}) - \dim N(T) = 4$, so $R(T) = M_{2\times2}(\mathbb{R})$. Thus we have shown that T is one-to-one and onto, and hence invertible.

3. Theorem 2.19 says that a pair of finite-dimensional vector spaces over the same field are isomorphic if and only if they have the same dimension. Thus the vector spaces in (b) and (c) are isomorphic, and the ones in (a) and (d) are not. To do (d), one could recall that the dimension of traceless $n \times n$ matrices is $n^2 - 1$, or just note that V is a subspace of $M_{2\times2}(\mathbb{R})$ but not all of $M_{2\times2}(\mathbb{R})$, so we must have $\dim V < 4$.

5. By definition, we have $AA^{-1} = A^{-1}A = I$. Taking transposes of these equations, and remembering that $(AB)^t = B^tA^t$, we get $(A^{-1})^tA^t = A^t(A^{-1})^t = I$. Thus the definition of inverse says that A^t is invertible and $(A^t)^{-1} = (A^{-1})^t$.

6. Multiplying both sides of $AB = 0$ on the left by A^{-1} yields $B = A^{-1}0 = 0$.

Section 2.5.

2. Recall (p.112) that if $\beta = \{x_1, \ldots, x_n\}$ and $\beta' = \{x'_1, \ldots, x'_n\}$ then the jth column of the change-of-coordinates matrix from β' to β is $[x'_j]_\beta$.

(a) Since $(a_1, a_2) = a_1e_1 + a_2e_2$, we have $[x'_1]_\beta = (a_1, a_2)$. The second column is similar, and we get the resulting change of coordinate matrix

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}.$$

(b) Since $(0, 10) = 4(-1, 3) + 2(2, -1)$ and $(5, 0) = (-1, 3) + 3(2, -1)$ we have $[x'_1]_\beta = (4, 2)$ and $[x'_2]_\beta = (1, 3)$. Thus we get the change of coordinate matrix

$$\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}.$$

(c) This could be solved by solving a linear system of equations as in (b), but we'll use a different method. From part (a), the change of coordinate matrix from β to β' (that is, in the opposite direction) is

$$\begin{pmatrix} 2 & -1 \\ 5 & -3 \end{pmatrix}.$$

By the remark on the top of p.112, the change of coordinates in the opposite direction is given by the inverse matrix, so we get an answer of

$$\begin{pmatrix} 2 & -1 \\ 5 & -3 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & -1 \\ 5 & -2 \end{pmatrix}.$$

(d) Solving $(2, 1) = a(-4, 3) + b(2, -1)$ yields $a = 2$, $b = 5$, so $[x'_1]_\beta = (2, 5)$. Similarly, we have $(-4, 1) = -(-4, 3) - 4(2, -1)$ so $[x'_2]_\beta = (-1, -4)$. Thus the desired change of coordinates matrix is

$$\begin{pmatrix} 2 & -1 \\ 5 & -4 \end{pmatrix}.$$

7.

(a) The idea is to find the matrix for T with respect to a basis for which this matrix is particularly simple. From there, we can use change of basis to find the matrix for T with respect to the standard basis, which will allow us to write a formula for T.

Since $(1, m)$ is on the axis of reflection, we want $T(1, m) = (1, m)$. The perpendicular line L' is the line $y = -m^{-1}x$, so $(-m, 1)$ is on this line. The image of that point under the reflection will be $(m, -1)$, so we want $T(-m, 1) = (m, -1)$. Thus if we let $\beta' = \{(1, m), (-m, 1)\}$ then

$$[T]_{\beta'} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

If β is the standard basis for $\mathbb{R}^2$, then by Exercise 2(a) the matrix that transforms β' coordinates into β coordinates is

$$Q = \begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix}.$$

By Theorem 2.23, we have

$$[T]_\beta = Q[T]_{\beta'}Q^{-1} = \begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{1+m^2} & \frac{m}{1+m^2} \\ \frac{-m}{1+m^2} & \frac{1}{1+m^2} \end{pmatrix} = \frac{1}{1+m^2} \begin{pmatrix} 1 - m^2 & 2m \\ 2m & m^2 - 1 \end{pmatrix}.$$

Since β is the standard basis, Theorem 2.15 says that T is multiplication by $[T]_\beta$. Thus

$$T(a,b) = [T]_\beta \begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{1+m^2} \begin{pmatrix} (1-m^2)a + 2bm \\ 2am + (m^2-1)b \end{pmatrix}.$$

(b) Let L and L' be as in part (a). We take for granted that $\mathbb{R}^2 = L \oplus L'$, so that it makes sense to talk about the projection of L along L'. Recall that every $x \in \mathbb{R}^2$ can be written uniquely as $x = x_1 + x_2$ with $x_1 \in L$ and $x_2 \in L'$, and that $T(x) = x_1$. Thus since $(1,m) \in L$ we have $T(1,m) = (1,m)$, and since $(-m,1) \in L'$ we have $T(-m,1) = 0$. Thus if β' is as before, we have

$$[T]_{\beta'} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

We are in exactly the same position as in part (a), with a slightly different $[T]_{\beta'}$. Proceeding as before, we compute

$$[T]_\beta = Q[T]_{\beta'}Q^{-1} = \frac{1}{1+m^2} \begin{pmatrix} 1 & m \\ m & m^2 \end{pmatrix}.$$

Using the reasoning from before, we get

$$T(a,b) = \frac{1}{1+m^2} \begin{pmatrix} a + bm \\ am + bm^2 \end{pmatrix}.$$

Section 2.6.

3.
(a) If $\beta^* = \{f_1, f_2, f_3\}$, the definition of dual basis says that

$$\begin{aligned} 1 &= f_1(e_1 + e_3) = f_1(e_1) + f_1(e_3) \\ 0 &= f_1(e_1 + 2e_2 + e_3) = f_1(e_1) + 2f_1(e_2) + f_1(e_3) \\ 0 &= f_1(e_3). \end{aligned}$$

So we have

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f_1(e_1) \\ f_1(e_2) \\ f_1(e_3) \end{pmatrix}.$$

Solving this matrix equation gives

$$f_1(e_1) = 1, \quad f_1(e_2) = -\frac{1}{2}, \quad f_1(e_3) = 0,$$

so that

$$f_1(x,y,z) = f_1(xe_1 + ye_2 + ze_3) = x - \frac{1}{2}y.$$

Proceeding as before, we get

$$\begin{aligned} 0 &= f_2(e_1 + e_3) = f_2(e_1) + f_2(e_3) \\ 1 &= f_2(e_1 + 2e_2 + e_3) = f_2(e_1) + 2f_2(e_2) + f_2(e_3) \\ 0 &= f_2(e_3), \end{aligned}$$

or

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f_2(e_1) \\ f_2(e_2) \\ f_2(e_3) \end{pmatrix}.$$

Solving this system gives

$$f_2(e_1) = 0, \quad f_2(e_2) = \frac{1}{2}, \quad f_2(e_3) = 0.$$

Hence $f_2(x, y, z) = \frac{1}{2}y$.

To find f_3, we proceed as before, and solve

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f_3(e_1) \\ f_3(e_2) \\ f_3(e_3) \end{pmatrix}.$$

This yields

$$f_3(e_1) = -1, \quad f_3(e_2) = 0, \quad f_3(e_3) = 1,$$

or $f_3(x, y, z) = z - x$. Thus the dual basis $\beta^* = \{f_1, f_2, f_3\}$ for the functions f_j defined above.

(b) As before, suppose $\beta^* = \{f_1, f_2, f_3\}$. Then we have

$$1 = f_1(1), \quad 0 = f_1(x), \quad 0 = f_1(x^2)$$

Thus $f_1(ax^2 + bx + c) = c$. Similarly,

$$0 = f_2(1), \quad 1 = f_2(x), \quad 0 = f_2(x^2)$$

so $f_2(ax^2 + bx + c) = b$. Repeating, we get $f_3(ax^3 + bx + c) = a$.

9. First suppose that T is linear, and we will prove that there exist $f_1, \dots, f_m \in (F^n)^*$ such that $T(x) = (f_1(x), \dots, f_m(x))$. Following the hint, let $\{e_1, \dots, e_m\}$ be the standard basis for F^m and let $\{g_1, \dots, g_m\}$ be its dual basis. That is, $g_i \in (F^m)^*$ and

$$g_i(e_j) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

As the hint instructs, we define $f_i(x) = g_i(T(x)) \in (F^n)^*$. We must now prove that $f_1, \dots, f_m$ have the required property.

Given $x \in F^n$, we can write

$$T(x) = (c_1, \dots, c_m) = c_1e_1 + \cdots + c_me_m$$

for some coefficients $c_j \in F$. We then have

$$f_i(x) = g_i(T(x)) = g_i(c_1e_1 + \cdots + c_me_m) = c_i$$

by the linearity of g_i and the definition of dual basis. Thus

$$(f_1(x), \dots, f_m(x)) = (c_1, \dots, c_m) = T(x).$$

Since x was arbitrary, we have shown that $T(x) = (f_1(x), \ldots, f_m(x))$ for all $x \in F^n$.

We must now prove the opposite direction. That is, assume that $T : F^n \to F^m$ is a function with the property that there exist $f_1, \ldots, f_m \in (F^n)^*$ such that $T(x) = (f_1(x), \ldots, f_m(x))$ for all $x \in F^n$. We will now show that T is linear. By the linearity of the f_i, we have for $x, y \in F^n$ and $c \in F$

$$\begin{aligned} T(x+cy) &= (f_1(x+cy), \ldots, f_m(x+cy)) \\ &= (f_1(x) + cf_1(y), \ldots, f_m(x) + cf_m(y)) \\ &= (f_1(x), \ldots, f_m(x)) + c(f_1(y), \ldots, f_m(y)) \\ &= T(x) + cT(y). \end{aligned}$$

Thus T is linear.

19. Following the hint, we expect to use Exercise 34 of Section 2.1. Since this was not previously completed on homework, we state the exercise as a lemma now, and then prove it at the end of the exercise.

Lemma (Exercise 2.1.34)**.** *Let V and W be vector spaces over a common field F, and let β be a basis for V. Then for every function $f : \beta \to W$ there exists a unique linear transformation $T : V \to W$ such that $T(x) = f(x)$ for all $x \in \beta$.*

Let β be a basis for W. By assumption $W \subset V$ but $W \neq V$, so we may choose some element $x_0 \in V$ with $x_0 \notin W$. Since $x_0 \notin \operatorname{span} \beta$, we have that $\{x_0\} \cup \beta$ is linearly independent. We can now extend $\{x_0\} \cup \beta$ to a basis β' of V by the corollary to Theorem 1.13. Define $f : \beta' \to F$ by

$$f(x) = \begin{cases} 1 & x = x_0 \\ 0 & x \neq x_0. \end{cases}$$

In particular, $f(x) = 0$ for $x \in \beta$. By the Lemma above, there exists a linear transformation $T : V \to F$ (i.e. $T \in V^*$) such that $T(x) = f(x)$ for $x \in \beta$. Since any $x \in W$ can be written as $c_1x_1 + \cdots c_nx_n$ with $x_j \in \beta$, we have

$$T(x) = c_1 f(x_1) + \cdots c_n f(x_n) = 0$$

for $x \in W$. But $T(x_0) = 1$, so $T \neq 0$. Thus T is the linear functional we wished to construct (called f in the problem description - the f here is a function, but not a linear one).

Proof of lemma. Define T as follows. For $x \in V$, there exists a unique subset $\{x_1, \ldots, x_n\} \subset \beta$ and unique non-zero scalars $c_1, \ldots c_n$ such that $x = c_1x_1 + \cdots + c_nx_n$. (The fact that the scalars are required to be non-zero lets us choose unique basis vectors x_j). Define

$$T(x) = c_1 f(x_1) + \cdots c_n f(x_n).$$

We then have $T(x) = x$ for all $x \in \beta$, but we must check linearity and uniqueness.

We defined $T(x)$ in terms of a particular representation of x in terms of basis vectors that have non-zero coefficients, but we now prove the fact that if we have *any* representation

$$x = c_1x_1 + \cdots + c_nx_n$$

with $x_j \in \beta$, then we have

$$T(x) = c_1 f(x_1) + \cdots + c_n f(x_n).$$

We will need that fact to prove that T is linear.

So suppose $\{x'_1, \ldots, x'_m\}$ is another subset of β such that there are coefficients $c'_i \in F$ such that

$$x = c'_1 x'_1 + \cdots + c'_m x'_m.$$

By the uniqueness of the representation of x, we must have $m \geq n$, and we can reorder $\{x'_1, \ldots, x'_m\}$ so that $x'_j = x_j$ and $c'_j = c_j$ for $1 \leq j \leq n$, and $c_j = 0$ for $j > n$. Thus

$$T(x) = c_1 f(x_1) + \cdots + c_n f(x_n) = c'_1 f(x'_1) + \cdots + c_m f(x'_m).$$

To show that T is linear, suppose $x, y \in V$ and $a \in F$. Then we can write

$$x = c_1 x_1 + \cdots + c_n x_n, \quad y = d_1 y_1 + \cdots + d_m y_m$$

for $x_i, y_j \in \beta$ and $c_i, d_j \in F$. Let

$$\{z_1, \ldots, z_\ell\} = \{x_1, \ldots, x_n\} \cup \{y_1, \ldots, y_m\}.$$

Then we can write

$$x = c'_1 z_1 + \cdots + c'_\ell z_\ell, \quad y = d'_1 z_1 + \cdots + d'_\ell z_\ell$$

by defining new coefficients to be zero, if necessary. Then by the preceding remark we can calculate

$$\begin{aligned}
T(x + ay) &= T((c'_1 + ad'_1)z'_1 + \cdots + (c'_\ell + ad'_\ell)z_\ell) \\
&= (c'_1 + ad'_1)f(z_1) + \cdots + (c'_\ell + ad'_\ell)f(z_\ell) \\
&= (c'_1 f(z_1) + \cdots + c'_\ell f(z_\ell)) + a(d'_1 f(z_1) + \cdots d'_\ell f(z_\ell)) \\
&= T(x) + dT(y)
\end{aligned}$$

Thus T is linear. To prove uniqueness, suppose T' is another linear transformation with the given property. Then $(T - T')(x) = 0$ for all $x \in \beta$. Since every element of V can be written as a linear combination of elements of β, it follows that $T - T' = 0$. $\square$

20.
(a) First suppose that T is onto. We we to show that T^t is one-to-one, so supose $T^t(f) = 0$ for some $f \in W^*$. We wish to show that $f = 0$. Unpacking the definition, $T^t(f) = 0$ implies that $f \circ T$ is the zero linear functional in V^*. That is, $f(T(x)) = 0$ for every $x \in V$. But for $y \in W$, we can write $y = T(x)$ for some $x \in V$. Then $f(y) = f(T(x)) = 0$. Since y was arbitrary, $f(y) = 0$ for all $y \in W$. Thus $f = 0 \in W^*$, as desired.

We now need to show that if T^t is one-to-one then T is onto. Suppose to a contradiction that $R(T) \neq W$. Then by Exercise 19, there is some $f \in W^*$ such that $f(x) = 0$ for all $x \in R(T)$, but $f \neq 0$. Let $g = (T^t)(f) \in V^*$. Then for $x \in V$, we have $g(x) = f(T(x)) = 0$ since $T(x) \in R(T)$. But x was arbitrary, so g is the zero linear functional. Thus $(T^t)(f) = 0$, but $f \neq 0$ by an earlier assumption. This contradicts the fact that T^t was assumed to be one-to-one. (For those who are interested, this proof was actually a proof by contraposition, not a proof by contradiction, for what that's worth).

(b) Suppose that T^t is onto, and suppose $x_0 \in V$ and $T(x_0) = 0$. We must show that $x_0 = 0$. Suppose to a contradiction that $x_0 \neq 0$. Then by the corollary to Theorem 1.13 we can choose a basis β for V with $x_0 \in \beta$. Define the function $\phi : \beta \to F$ by

$$\phi(x) = \begin{cases} 1 & x = x_0 \\ 0 & x \neq x_0. \end{cases}$$

By Exercise 19, there exists a non-zero $f \in V^*$ such that $f(x) = \phi(x)$ for $x \in \beta$. Since T^t is onto, there is some $g \in W^*$ such that $(T^t)(g) = f$. That is, $g(T(x)) = f(x)$ for all $x \in V$. Now, if $x \in \beta$ and $x \neq x_0$, then $\phi(x) = 0$ so

$$f(x) = g(T(x)) = g(\phi(x)) = g(0) = 0.$$

On the other hand

$$f(x_0) = g(T(x_0)) = 0$$

since we assumed $T(x_0) = 0$. But we have proven that $f(x) = 0$ for every $x \in \beta$, and since β is a basis for V we can conclude that $f = 0$. But this contradicts the assumption that $f \neq 0$, so we must have had that $x_0 = 0$ and T is one-to-one.

Now suppose that T is one-to-one, and we will show that T^t is onto. Let $g \in V^*$, and we will show there exists $f \in W^*$ with $(T^t)(f) = g$. That is, we must find f with $f(T(x)) = g(x)$ for all $x \in V$. We know how to define f on $R(T)$, but to define f on all of W it will help to have the following lemma (as before, the proof is at the end of the problem).

Lemma. *If $W_1 \subseteq W$ is a subspace, then there exists a subspace W_2 of V for which $W = W_1 \oplus W_2$.*

Let W_2 be a subspace of W for which $W = R(T) \oplus W_2$. Given $w \in W$, we can write it uniquely as $w_1 + w_2$ where $w_1 \in R(T)$ and $w_2 \in W_2$. Since T is one-to-one, given $y \in R(T)$ there exists a unique $x \in V$ with $T(x) = y$. Thus given $w \in W$, write it as $w_1 + w_2$ as above, and let $f(w) = g(T(x))$, where $x \in V$ is the unique element with $T(x) = w_1$. We now have $g(x) = f(T(x))$ for all $x \in V$, and $f = (T^t)(g)$ once we show that f is linear.

Given $c \in F$ and $w \in W$, we have $f(cw) = f(cw_1 + cw_2)$ where we have decomposed w as above. Then if x is the unique element of V with $T(x) = w_1$, we have $T(cx) = cw_1$, so

$$f(cw) = g(cx) = cg(x) = cf(w).$$

Now if $w, w' \in W$ decompose as $w_1 + w_2$ and $w_1' + w_2'$ respectively, and we have unique x, x' with $T(x) = w_1$ and $T(x') = w_1'$, then $T(x + x') = w_1 + w_\prime$. Thus, the definitions tell us

$$f(w + w') = f((w_1 + w_1') + (w_2 + w_2')) = g(x + x') = g(x) + g(x') = f(w_1 + w_2) + f(w_1' + w_2').$$

Proof of lemma. This was Exercise 1.6.34(a), except we have not assumed that W is finite-dimensional. However, the proof is exactly the same as the exercise, but one must use the corollary of Theorem 1.13 from Section 1.7 to extend linearly independent sets to bases for infinite dimensional spaces. □