Solutions to Homework #2.

Section 1.5.

3. Denote the 5 vectors by $v_1, \ldots, v_5$ and note that $v_1 + v_2 + v_3 - v_5 = v_4$.

9. Say $\{u, v\}$ is linearly dependent. Then there exist nonzero elements $a, b \in F$ such that $au+bv = 0$. This then yields $u = (-ba^{-1})v$.

Assume u is a multiple of v. Then for some nonzero $c \in F$ we have $u = cv$. This implies that $u + (-c)v = 0$ so the two vectors are linearly dependent.

11. First note that any two vectors v, w from the span of S look like

$$\begin{aligned} v &= a_1 u_1 + \ldots a_n u_n \\ w &= b_1 u_1 + \ldots b_n u_n \end{aligned}$$

for some $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{Z}_2$. Since S consists of linearly independent vectors the two vectors v, w are equal if and only if $a_1 = b_1, \ldots, a_n = b_n$. Therefore, the span of S has the same cardinality as the set $\{(a_1, \ldots, a_n) \mid a_1, \ldots, a_n \in \mathbb{Z}_2\}$. Since we work with $\mathbb{Z}_2$ the only elements of our field are $\{0, 1\}$. As a result there are exactly 2^n vectors in the span of S.

13. (a) Say $\{u, v\}$ is linearly independent and assume $a(u + v) + b(u - v) = 0$ for some $a, b \in F$. Then $(a + b)u + (a - b)v = 0$. By the linear independence of u, v we need

$$\begin{aligned} a + b &= 0 \\ a - b &= 0. \end{aligned}$$

Adding the two equations we get $2a = 0$ which implies $a = 0$ (since the characteristic of F is not 2). This then yields $b = 0$.

Say $\{u + v, u - v\}$ is linearly independent. Assume $cu + dv = 0$ for some $c, d \in F$. Note that $u = \frac{1}{2}(u + v + u - v), v = \frac{1}{2}(u + v - (u - v))$ (2 has an inverse because the characteristic is not 2). Then $cu + dv = 0$ becomes $c\frac{1}{2}(u + v + u - v) + d\frac{1}{2}(u + v - (u - v)) = 0$ and by independence and multiplication by 2

$$\begin{aligned} c + d &= 0 \\ c - d &= 0. \end{aligned}$$

which yields $c = d = 0$ just like above.

(b) For this the argument is similar to part (a) above, where one notes that

$$\begin{aligned} u &= \frac{1}{2}(u + v + u + w - (v + w)) \\ v &= \frac{1}{2}(u + v + v + w - (u + w)) \\ w &= \frac{1}{2}(u + w + v + w - (u + v)). \end{aligned}$$

Section 1.6.

3. No, the dimension of P_3 is 4 so it cannot be spanned by 3 vectors.

9. Let $\{e_1, e_2, e_3, e_4\}$ be the canonical basis for F^4. We can write

$$\begin{aligned} e_1 &= u_1 - u_2 \\ e_2 &= u_2 - u_3 \\ e_3 &= u_3 - u_4 \\ e_4 &= u_4. \end{aligned}$$

Therefore, an arbitrary vector (a, b, c, d) in F^4 can be written as

$$ae_1 + be_2 + ce_3 + de_4 = au_1 + (-a + b)u_2 + (-b + c)u_3 + (-c + d)u_4.$$

13. Subtracting the two equations yields $x_1 - x_2 = 0$ so $x_1 = x_2$. Then, any of the two equations implies $x_1 = x_2 = x_3$. A basis of the subspace of solutions therefore is $(1, 1, 1)$.

17. For $i > j$ let A_{ij} be the $n \times n$ matrix which has a +1 on row i, column j, and a -1 on row j, column i. These matrices are clearly linearly independent and they span the set of skew-symmetric matrices. There are $(n^2 - n)/2$ such matrices so this number is also the dimension of W.

20. (a) V has finite dimension n and is spanned by S. There exists a basis $\{v_1, \ldots, v_n\}$ of V. Since V is the span of S every element from $\{v_1, \ldots, v_n\}$ is a finite linear combination of vectors from S. Thus, there exists $\tilde{S} \subset S$, $\tilde{S}$ finite such that $\{v_1, \ldots, v_n\} \subset \text{span}(\tilde{S})$. As a result $V = \text{span}(\tilde{S})$. By theorem 1.9 from the book a subset of $\tilde{S}$ (and thus a subset of S) is a basis for V.

(b) By part (a) S contains a basis of V. A basis of V has exactly n elements so we are done.

26. $\dim(P_n) = n + 1$. Note that $\{f \in P_n(\mathbb{R}) : f(a) = 0\} = \{(x - a)g \mid g \in P_{n-1}\}$. This subset can therefore be identified with P_{n-1} so the dimension is n.

29. (a) Following the hint, let $\{u_1, \ldots, u_k\}$ be a basis for $W_1 \cap W_2$ and extend it to a basis $\{u_1, \ldots, u_k, v_1, \ldots, v_m\}$ for W_1 and to a basis $\{u_1, \ldots, u_k, w_1, \ldots, w_p\}$ for W_2. Then, a basis for $W_1 + W_2$ will be $\{u_1, \ldots, u_k, v_1, \ldots, v_m, w_1, \ldots, w_p\}$. As a result

$$\begin{aligned} \dim(W_1 \cap W_2) &= k \\ \dim(W_1) &= k + m \\ \dim(W_2) &= k + p \\ \dim(W_1 + W_2) &= k + m + p \end{aligned}$$

Therefore, $\dim(W_1 \cap W_2 = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$.

(b) Say $V = W_1 \oplus W_2$. Then $W_1 \cap W_2 = \{0\}$ so $\dim(W_1 \cap W_2) = 0$ and the formula from part (a) yields $\dim(V) = \dim(W_1 \cap W_2 = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2) = \dim(W_1) + \dim(W_2) - 0 = \dim(W_1) + \dim(W_2)$.

Say $\dim(V) = \dim(W_1) + \dim(W_2)$. This implies, by the above formula, that $\dim(W_1 \cap W_2) = 0$ so $W_1 \cap W_2 = \{0\}$ and $V = W_1 \oplus W_2$.

34. (a) Let $\{v_1, \dots, v_m\}$ be a basis for W_1. If $V = W_1$ set $W_2 = \{0\}$. Otherwise, extend the basis of W_1 to a basis $\{v_1, \dots, v_m, w_1, \dots, w_n\}$ of V. Let $W_2 = \operatorname{span}\{w_1, \dots, w_n\}$. It is then immediate that $V = W_1 \oplus W_2$.

(b) $W_1 = \{(0, a_2) \mid a_2 \in \mathbb{R}\}$ and $W_2 = \{(a_2/2, a_2) \mid a_2 \in \mathbb{R}\}$.