Solutions to Homework #2.

Section 1.5.

3. Denote the 5 vectors by \( v_1, \ldots, v_5 \) and note that \( v_1 + v_2 + v_3 - v_5 = v_4 \).

9. Say \( \{u, v\} \) is linearly dependent. Then there exist nonzero elements \( a, b \in F \) such that \( au + bv = 0 \). This then yields \( u = (-ba^{-1})v \).

   Assume \( u \) is a multiple of \( v \). Then for some nonzero \( c \in F \) we have \( u = cv \). This implies that \( u + (-c)v = 0 \) so the two vectors are linearly dependent.

11. First note that any two vectors \( v, w \) from the span of \( S \) look like

\[
\begin{align*}
v &= a_1 u_1 + \ldots + a_n u_n \\
w &= b_1 u_1 + \ldots + b_n u_n
\end{align*}
\]

for some \( a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{Z}_2 \). Since \( S \) consists of linearly independent vectors the two vectors \( v, w \) are equal if and only if \( a_1 = b_1, \ldots, a_n = b_n \). Therefore, the span of \( S \) has the same cardinality as the set \( \{(a_1, \ldots, a_n) \mid a_1, \ldots, a_n \in \mathbb{Z}_2\} \). Since we work with \( \mathbb{Z}_2 \) the only elements of our field are \( \{0, 1\} \). As a result there are exactly \( 2^n \) vectors in the span of \( S \).

13. (a) Say \( \{u, v\} \) is linearly independent and assume \( a(u + v) + b(u - v) = 0 \) for some \( a, b \in F \). Then \( (a + b)u + (a - b)v = 0 \). By the linear independence of \( u, v \) we need

\[
\begin{align*}
a + b &= 0 \\
a - b &= 0
\end{align*}
\]

Adding the two equations we get \( 2a = 0 \) which implies \( a = 0 \) (since the characteristic of \( F \) is not 2). This then yields \( b = 0 \).

Say \( \{u + v, u - v\} \) is linearly independent. Assume \( cu + dv = 0 \) for some \( c, d \in F \). Note that \( u = \frac{1}{2}(u + v + u - v), v = \frac{1}{2}(u + v - (u - v)) \) (2 has an inverse because the characteristic is not 2). Then \( cu + dv = 0 \) becomes \( c_2(u + v + u - v) + d_2(u + v - (u - v)) = 0 \) and by independence and multiplication by 2

\[
\begin{align*}
c + d &= 0 \\
c - d &= 0
\end{align*}
\]

which yields \( c = d = 0 \) just like above.

(b) For this the argument is similar to part (a) above, where one notes that

\[
\begin{align*}
u &= \frac{1}{2}(u + v + u + w - (v + w)) \\
v &= \frac{1}{2}(u + v + v + w - (u + w)) \\
w &= \frac{1}{2}(u + w + v + w - (u + v))
\end{align*}
\]
Section 1.6.

3. No, the dimension of $P_3$ is 4 so it cannot be spanned by 3 vectors.

9. Let $\{e_1,e_2,e_3,e_4\}$ be the canonical basis for $F^4$. We can write

$$
e_1 = u_1 - u_2
$$
$$
e_2 = u_2 - u_3
$$
$$
e_3 = u_3 - u_4
$$
$$
e_4 = u_4.
$$

Therefore, an arbitrary vector $(a, b, c, d)$ in $F^4$ can be written as

$$
ae_1 + be_2 + ce_3 + de_4 = au_1 + (-a + b)u_2 + (-b + c)u_3 + (-c + d)u_4.
$$

13. Subtracting the two equations yields $x_1 - x_2 = 0$ so $x_1 = x_2$. Then, any of the two equations implies $x_1 = x_2 = x_3$. A basis of the subspace of solutions therefore is $(1,1,1)$.

17. For $i > j$ let $A_{ij}$ be the $n \times n$ matrix which has a $+1$ on row $i$, column $j$, and a $-1$ on row $j$, column $i$. These matrices are clearly linearly independent and they span the set of skew-symmetric matrices. There are $(n^2 - n)/2$ such matrices so this number is also the dimension of $W$.

20. (a) $V$ has finite dimension $n$ and is spanned by $S$. There exists a basis $\{v_1, \ldots, v_n\}$ of $V$. Since $V$ is the span of $S$ every element from $\{v_1, \ldots, v_n\}$ is a finite linear combination of vectors from $S$. Thus, there exists $\tilde{S} \subset S$, $\tilde{S}$ finite such that $\{v_1, \ldots, v_n\} \subset \text{span}(\tilde{S})$. As a result $V = \text{span}(\tilde{S})$. By theorem 1.9 from the book a subset of $\tilde{S}$ (and thus a subset of $S$) is a basis for $V$.

(b) By part (a) $S$ contains a basis of $V$. A basis of $V$ has exactly $n$ elements so we are done.

26. $\dim(P_n) = n + 1$. Note that $\{f \in P_n(\mathbb{R}) : f(a) = 0\} = \{(x - a)g \mid g \in P_{n-1}\}$. This subset can therefore be identified with $P_{n-1}$ so the dimension is $n$.

29. (a) Following the hint, let $\{u_1, \ldots, u_k\}$ be a basis for $W_1 \cap W_2$ and extend it to a basis $\{u_1, \ldots, u_k, v_1, \ldots, v_m\}$ for $W_1$ and to a basis $\{u_1, \ldots, u_k, w_1, \ldots, w_p\}$ for $W_2$. Then, a basis for $W_1 + W_2$ will be $\{u_1, \ldots, u_k, v_1, \ldots, v_m, w_1, \ldots, w_p\}$. As a result

$$
dim(W_1 \cap W_2) = k
$$
$$
dim(W_1) = k + m
$$
$$
dim(W_2) = k + p
$$
$$
dim(W_1 + W_2) = k + m + p
$$

Therefore, $\dim(W_1 \cap W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$.

(b) Say $V = W_1 \oplus W_2$. Then $W_1 \cap W_2 = \{0\}$ so $\dim(W_1 \cap W_2) = 0$ and the formula from part (a) yields $\dim(V) = \dim(W_1 \cap W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2) = \dim(W_1) + \dim(W_2) - 0 = \dim(W_1) + \dim(W_2)$.

Say $\dim(V) = \dim(W_1) + \dim(W_2)$. This implies, by the above formula, that $\dim(W_1 \cap W_2) = 0$ so $W_1 \cap W_2 = \{0\}$ and $V = W_1 \oplus W_2$. 

2
34. (a) Let \( \{v_1, \ldots, v_m\} \) be a basis for \( W_1 \). If \( V = W_1 \) set \( W_2 = \{0\} \). Otherwise, extend the basis of \( W_1 \) to a basis \( \{v_1, \ldots, v_m, w_1, \ldots, w_n\} \) of \( V \). Let \( W_2 = \text{span}\{w_1, \ldots, w_n\} \). It is then immediate that \( V = W_1 \oplus W_2 \).

(b) \( W_1 = \{(0, a_2) \mid a_2 \in \mathbb{R}\} \) and \( W_2 = \{(a_2/2, a_2) \mid a_2 \in \mathbb{R}\} \).