Solutions to Homework #1.

Section 1.2.

6. \[ M = \begin{pmatrix} 4 & 2 & 1 & 3 \\ 5 & 1 & 1 & 4 \\ 3 & 1 & 2 & 6 \end{pmatrix} \]

By definition \((2M)_{ij} = 2M_{ij}\) so

\[ 2M = \begin{pmatrix} 8 & 4 & 2 & 6 \\ 10 & 2 & 2 & 8 \\ 6 & 2 & 4 & 12 \end{pmatrix} \]

The matrix \(2M - A\) records the goods sold during the June sale.

\[ 2M - A = \begin{pmatrix} 8 & 4 & 2 & 6 \\ 10 & 2 & 2 & 8 \\ 6 & 2 & 4 & 12 \end{pmatrix} - \begin{pmatrix} 5 & 3 & 1 & 2 \\ 6 & 2 & 1 & 5 \\ 1 & 0 & 3 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 1 & 4 \\ 4 & 0 & 1 & 3 \\ 5 & 2 & 1 & 9 \end{pmatrix} \]

The total number of suites of all types sold is the sum of all entries

\[ \sum_{i,j} (2M - A)_{ij} = 34 \]

16. Yes. The axioms are satisfied for all elements of \(\mathbb{R}\) and hence in particular for all elements of \(\mathbb{Q}\). (In fact, more generally, for any vector space \(V\) over a field \(F\), if another field \(F'\) sits inside \(F\) compatibly with its field operations, then \(V\) is tautologically a vector space over \(F'\) as well.)

21. We must check \(Z\) satisfies axioms (VS1)-(VS8) using the fact that \(V,W\) satisfy them. (Be sure you understand the justification for each equality in the lines below.)

(VS1) \((v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) = (v_2 + v_1, w_2 + w_1) = (v_2, w_2) + (w_1, v_1)\).

(VS2) \(((v_1, w_1) + (v_2, w_2)) + (v_3, w_3) = (v_1 + v_2, w_1 + w_2) + (v_3, w_3) = ((v_1 + v_2) + v_3, (w_1 + w_2) + w_3) = (v_1 + (v_2 + v_3), w_1 + (w_2 + w_3)) = (v_1, w_1) + (v_2 + v_3, w_2 + w_3) = (v_1, w_1) + ((v_2, w_2) + (v_3, w_3))\)

(V3) Take 0 = (0, 0). Then \((v, w) + (0, 0) = (v + 0, w + 0) = (v, w)\).

(V4) Given \((v, w)\), use \((-v, -w)\). Then \((v, w) + (-v, -w) = (v + (-v), w + (-w)) = (0, 0)\).

(V5) \(1(v, w) = (1v, 1w) = (v, w)\).

(V6) \((ab)(v, w) = ((ab)v, (ab)w) = a(bv, aw) = a(bv, aw)\).

(V7) \(a((v_1, w_1) + (v_2, w_2)) = a(v_1 + v_2, w_1 + w_2) = a(v_1 + v_2), a(w_1 + w_2) = (av_1 + av_2, aw_1 + aw_2) = (av_1 + aw_1) + (av_2, aw_2) = a(v_1, w_1) + a(v_2, w_2)\).

(V8) \((a + b)(v, w) = ((a + b)v, (a + b)w) = (av + bv, aw + bw) = (av, aw) + (bv, bw) = a(v, w) + b(v, w)\).

Section 1.3.

8. (a) Yes. The equations are linear: the addition or scaling of solutions is still a solution.

(b) No. For example, 0 is not contained in \(W_2\).

(c) Yes. The equation is linear: the addition or scaling of solutions is still a solution.
are vectors $W_9$. (1) subspace. $W$ $a \in W$. But this contradicts our assumption that $w \neq 0 \in W$.

9. $W_1 \cap W_3 = \{0\}$.
$W_1 \cap W_4 = W_1$ which is the line \{(3t, t, -t) | t \in \mathbb{R}\}.
$W_3 \cap W_4$ is the line \{(11t, 3t, -t) | t \in \mathbb{R}\}.

19. Suppose $W_1 \subset W_2$. Then $W_1 \cup W_2 = W_2$ and so $W_1 \cup W_2$ is a subspace since $W_2$ is a subspace. Similarly, suppose $W_2 \subset W_1$. Then $W_1 \cup W_2 = W_1$ and so $W_1 \cup W_2$ is a subspace since $W_1$ is a subspace.

Conversely, suppose $W_1 \cup W_2$ is a subspace. Suppose also that $W_1 \not\subset W_2$, $W_2 \not\subset W_1$, so there are vectors $w_1 \in W_1, w_2 \in W_2$ such that $w_1 \not\in W_2, w_2 \not\in W_1$. Then $W_1 \cup W_2$ must contain $w_1 + w_2$ since it is a subspace and contains each of $w_1$ and $w_2$. But then $w_1 + w_2$ must be in $W_1$ or $W_2$ since it is in their union. If $w_1 + w_2 \in W_1$, then $w_2 = (w_1 + w_2) - w_1 \in W_2$ since $W_2$ is a subspace. But this contradicts our assumption that $w_2 \not\in W_1$. Similarly, if $w_1 + w_2 \in W_2$, then $w_1 = (w_1 + w_2) - w_2 \in W_1$ since $W_1$ is a subspace. But this contradicts our assumption that $w_1 \not\in W_2$. Thus either $W_1 \subset W_2$ or $W_2 \subset W_1$.

23. (a) We will verify the three conditions of Theorem 1.3.
(i) Since $W_1$ and $W_2$ are subspaces, 0 is in $W_1$ and $W_2$, thus $0 = 0 + 0 \in W_1 + W_2$
(ii) Given $v_1, w_1 \in W_1, v_2, w_2 \in W_2$, consider $v_1 + v_2, w_1 + w_2 \in W_1 + W_2$. Then we have $(v_1 + v_2) + (w_1 + w_2) = (v_1 + w_1) + (v_2 + w_2) \in W_1 + W_2$ since $v_1 + w_1 \in W_1, v_2 + w_2 \in W_2$.
(iii) Given $w_1 \in W_1, w_2 \in W_2$, consider $w_1 + w_2 \in W_1 + W_2$. Then for any $a \in F$, we have $a(w_1 + w_2) = (aw_1) + (aw_2) \in W_1 + W_2$.

(b) $W_1 \subset W_1 + W_2$ since $0 \in W_2$ and so $w_1 + 0 \in W_1 + W_2$ for any $w_1 \in W_1$. Similarly, $W_2 \subset W_1 + W_2$ since $0 \in W_1$ and so $0 + w_2 \in W_1 + W_2$ for any $w_2 \in W_2$.

29. We must show $W_1 + W_2 = M_{n \times n}(F)$ and $W_1 \cap W_2 = \{0\}$.
Given $A \in M_{n \times n}(F)$, let $A_u, A_d, A_t \in M_{n \times n}(F)$ denote its respective strictly upper triangular, diagonal, and strictly lower triangular parts. To be more precise, their entries are given by

\[(A_u)_{ij} = \begin{cases} A_{ij} & \text{if } i < j \\ 0 & \text{if } i \geq j \end{cases}, \quad (A_d)_{ij} = \begin{cases} A_{ij} & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}, \quad (A_t)_{ij} = \begin{cases} A_{ij} & \text{if } i > j \\ 0 & \text{if } i \leq j \end{cases}\]

Thus we have $A = A_u + A_d + A_t$ and $(A^t)_u = (A_t)^t$, $(A^t)_d = A_d$, $(A^t)_t = (A_u)^t$.

Observe that $W_1$ contains $A_u + A_d + A_t$. Observe that $W_2$ contains $A_t$ and $A_u$, hence also the difference $A_t - A_u$. Since $A = (A_u + A_d + A_u) + (A_t - A_u)$, we see that $W_1 + W_2 = M_{n \times n}(F)$.

On the other hand, if $A \in W_1 \cap W_2$, then $A = A_t, A_d = 0, A_u = 0$, but also $A = A^t$ so $(A_t)^t = A_u = 0$. Thus $A = 0$ and so $W_1 \cap W_2 = \{0\}$.

Note: the book’s assumption that $\text{char } F \neq 2$ is a red herring and is not needed!

30. Suppose $V = W_1 \oplus W_2$. Let $v \in V$. Since $V = W_1 + W_2$, there exist $w_1 \in W_1, w_2 \in W_2$ such that $v = w_1 + w_2$. Suppose there were another $w'_1 \in W_1, w'_2 \in W_2$ such that $v = w'_1 + w'_2$. Then we have $w_1 + w_2 = v = w'_1 + w'_2$ and so $w_1 - w'_1 = w'_2 - w_2$. But since $W_1 \cap W_2 = \{0\}$, we have $w_1 - w'_1 = 0 = w'_2 - w_2$ and so $w_1 = w'_1, w_2 = w'_2$.
Conversely, suppose for each \( v \in V \), there exist unique \( w_1 \in W_1, w_2 \in W_2 \) such that \( v = w_1 + w_2 \). Then we immediately have \( V = W_1 + W_2 \). Now suppose \( w \in W_1 \cap W_2 \). Then we can write \( w = w + 0 \) or alternatively \( w = 0 + w \). Thus we must have \( w = 0 \) and so \( W_1 \cap W_2 = \{0\} \).

Section 1.4.

5. (a) Yes: \((2, -1, 1) = (1, 0, 2) - (-1, 1, 1)\).
   (b) No.
   (c) No.
   (d) Yes: \((2, -1, 1, -3) = 2(1, 0, 1, -1) - (0, 1, 1, 1)\).
   (e) Yes: \(-x^3 + 2x^2 + 3x + 3 = -1(x^3 + x^2 + x + 1) + 3(x^2 + x + 1) + (x + 1)\).
   (f) No.
   (g) Yes:
   \[
   \begin{pmatrix}
   1 & 2 \\
   -3 & 4
   \end{pmatrix}
   = 3
   \begin{pmatrix}
   1 & 0 \\
   -1 & 0
   \end{pmatrix}
   + 4
   \begin{pmatrix}
   0 & 1 \\
   0 & 1
   \end{pmatrix}
   + 2
   \begin{pmatrix}
   1 & 1 \\
   0 & 0
   \end{pmatrix}
   \]
   (h) No.

15. Suppose \( v \in \text{span}(S_1 \cap S_2) \), so \( v = a_1v_1 + \cdots + a_kv_k \), for vectors \( v_1, \ldots, v_k \in S_1 \cap S_2 \) and scalars \( a_1, \ldots, a_k \in F \). Then clearly \( v_1, \ldots, v_k \in S_1 \) so \( v \in \text{span}(S_1) \), and similarly, \( v_1, \ldots, v_k \in S_2 \) so \( v \in \text{span}(S_2) \). Thus \( v \in \text{span}(S_1) \cap \text{span}(S_2) \).

   Example when \( \text{span}(S_1 \cap S_2) = \text{span}(S_1) \cap \text{span}(S_2) \): take \( V = S_1 = S_2 \).

   Example when \( \text{span}(S_1 \cap S_2) \neq \text{span}(S_1) \cap \text{span}(S_2) \): take \( V = \mathbb{R}, S_1 = \{1\}, S_2 = \{2\} \).

Additional problem: Prove that every field \( F \) contains either the field of rational numbers \( \mathbb{Q} \) or a finite field \( \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z} \) for a prime \( p \).

Solution: Take the smallest subset of \( F \) containing 0, 1 and closed under addition, multiplication, and forming additive and multiplicative inverses. Some notation used in the following argument to justify the above strategy: Given a positive integer \( m \), we will write \([m] \in F \) for the sum of \( m \) copies \( 1 + \cdots + 1 \) of the multiplicative unit \( 1 \in F \). We will also write \([−m] \in F \) for the additive inverse \( −[m] \).

Suppose first that \( \text{char} F = 0 \) so that \([n] \neq 0 \) for any \( n > 0 \). Consider all elements of the form \([m][−1]\) in \( F \), for integers \( m, n \) with \( n \neq 0 \). From the field axioms, we see that they form a subset closed under addition, multiplication and forming additive and multiplicative inverses. We claim such elements are distinct whenever the corresponding rational numbers are distinct and hence form a copy of \( \mathbb{Q} \). From the field axioms, it suffices to see that any \([m][−1] \in F \) equal to 0 is in fact of the form \([0][−1]\). But if \([m][−1] = 0 \) then \([m] = [m][−1][n] = 0 \) so \([m] = 0 \) so \( m = 0 \) since \( \text{char} F = 0 \).

Now if \( \text{char} F \neq 0 \), then we have seen it is equal to some prime \( p \). Consider the \( p \) elements of the form \([m] \in F \), for integers \( m \). From the field axioms, we see that they form a field and in fact a copy of \( \mathbb{F}_p \).