Solutions to Homework #1.

Section 1.2.

6.

$$M = \begin{pmatrix} 4 & 2 & 1 & 3 \\ 5 & 1 & 1 & 4 \\ 3 & 1 & 2 & 6 \end{pmatrix}$$

By definition $(2M)_{ij} = 2M_{ij}$ so

$$2M = \begin{pmatrix} 8 & 4 & 2 & 6 \\ 10 & 2 & 2 & 8 \\ 6 & 2 & 4 & 12 \end{pmatrix}$$

The matrix $2M - A$ records the goods sold during the June sale.

$$2M - A = \begin{pmatrix} 8 & 4 & 2 & 6 \\ 10 & 2 & 2 & 8 \\ 6 & 2 & 4 & 12 \end{pmatrix} - \begin{pmatrix} 5 & 3 & 1 & 2 \\ 6 & 2 & 1 & 5 \\ 1 & 0 & 3 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 1 & 4 \\ 4 & 0 & 1 & 3 \\ 5 & 2 & 1 & 9 \end{pmatrix}$$

The total number of suites of all types sold is the sum of all entries

$$\sum_{i,j} (2M - A)_{ij} = 34$$

16. Yes. The axioms are satisfied for all elements of $\mathbb{R}$ and hence in particular for all elements of $\mathbb{Q}$. (In fact, more generally, for any vector space V over a field F, if another field F' sits inside F compatibly with its field operations, then V is tautologically a vector space over F' as well.)

21. We must check Z satisfies axioms (VS1)-(VS8) using the fact that V, W satisfy them. (Be sure you understand the justification for each equality in the lines below.)

(VS1) $(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) = (v_2 + v_1, w_2 + w_1) = (v_2, w_2) + (w_1, v_1)$.

(VS2) $((v_1, w_1) + (v_2, w_2)) + (v_3, w_3) = (v_1 + v_2, w_1 + w_2) + (v_3, w_3) = ((v_1 + v_2) + v_3, (w_1 + w_2) + w_3) = (v_1 + (v_2 + v_3), w_1 + (w_2 + w_3)) = (v_1, w_1) + (v_2 + v_3, w_2 + w_3) = (v_1, w_1) + ((v_2, w_2) + (v_3, w_3))$

(V3) Take $0 = (0, 0)$. Then $(v, w) + (0, 0) = (v + 0, w + 0) = (v, w)$.

(V4) Given (v, w), use $(-v, -w)$. Then $(v, w) + (-v, -w) = (v + (-v), w + (-w)) = (0, 0)$.

(V5) $1(v, w) = (1v, 1w) = (v, w)$.

(V6) $(ab)(v, w) = ((ab)v, (ab)w) = (a(bv), a(bw)) = a(bv, bw) = a(b(v, w))$.

(V7) $a((v_1, w_1) + (v_2, w_2)) = a(v_1 + v_2, w_1, w_2) = (a(v_1 + v_2), a(w_1 + w_2)) = (av_1 + av_2, aw_1 + aw_2) = (av_1, aw_1) + (av_2, aw_2) = a(v_1, w_1) + a(v_2, w_2)$.

(V8) $(a + b)(v, w) = ((a + b)v, (a + b)w) = (av + bv, aw + bw) = (av, aw) + (bv, bw) = a(v, w) + b(v, w)$.

Section 1.3.

8. (a) Yes. The equations are linear: the addition or scaling of solutions is still a solution.

(b) No. For example, 0 is not contained in W_2.

(c) Yes. The equation is linear: the addition or scaling of solutions is still a solution.

(d) Yes. The equation is linear: the addition or scaling of solutions is still a solution.

(e) No. For example, 0 is not contained in W_5.

(f) No. While W_6 is preserved by scaling, it is not preserved by addition. For example, $v = (1, \sqrt{5/3}, 0)$ and $w = (0, \sqrt{2}, 1)$ are solutions, but $v + w = (1, \sqrt{5/3} + \sqrt{2}, 1)$ is not.

9. $W_1 \cap W_3 = \{0\}$.
$W_1 \cap W_4 = W_1$ which is the line $\{(3t, t, -t) | t \in \mathbb{R}\}$.
$W_3 \cap W_4$ is the line $\{(11t, 3t, -t) | t \in \mathbb{R}\}$.

19. Suppose $W_1 \subset W_2$. Then $W_1 \cup W_2 = W_2$ and so $W_1 \cup W_2$ is a subspace since W_2 is a subspace. Similarly, suppose $W_2 \subset W_1$. Then $W_1 \cup W_2 = W_1$ and so $W_1 \cup W_2$ is a subspace since W_1 is a subspace.

Conversely, suppose $W_1 \cup W_2$ is a subspace. Suppose also that $W_1 \not\subset W_2$, $W_2 \not\subset W_1$, so there are vectors $w_1 \in W_1, w_2 \in W_2$ such that $w_1 \notin W_2, w_2 \notin W_1$. Then $W_1 \cup W_2$ must contain $w_1 + w_2$ since it is a subspace and contains each of w_1 and w_2. But then $w_1 + w_2$ must be in W_1 or W_2 since it is in their union. If $w_1 + w_2 \in W_1$, then $w_2 = (w_1 + w_2) - w_1 \in W_1$ since W_1 is a subspace. But this contradicts our assumption that $w_2 \notin W_1$. Similarly, if $w_1 + w_2 \in W_2$, then $w_1 = (w_1 + w_2) - w_2 \in W_2$ since W_2 is a subspace. But this contradicts our assumption that $w_1 \notin W_2$. Thus either $W_1 \subset W_2$ or $W_2 \subset W_1$.

23. (a) We will verify the three conditions of Theorem 1.3.

(i) Since W_1 and W_2 are subspaces, 0 is in W_1 and W_2, thus $0 = 0 + 0 \in W_1 + W_2$

(ii) Given $v_1, w_1 \in W_1, v_2, w_2 \in W_2$, consider $v_1 + v_2, w_1 + w_2 \in W_1 + W_2$. Then we have $(v_1 + v_2) + (w_1 + w_2) = (v_1 + w_1) + (v_2 + w_2) \in W_1 + W_2$ since $v_1 + w_1 \in W_1, v_2 + w_2 \in W_2$.

(iii) Given $w_1 \in W_1, w_2 \in W_2$, consider $w_1 + w_2 \in W_1 + W_2$. Then for any $a \in F$, we have $a(w_1 + w_2) = (aw_1) + (aw_2) \in W_1 + W_2$.

(b) $W_1 \subset W_1 + W_2$ since $0 \in W_2$ and so $w_1 + 0 \in W_1 + W_2$ for any $w_1 \in W_1$. Similarly, $W_2 \subset W_1 + W_2$ since $0 \in W_1$ and so $0 + w_2 \in W_1 + W_2$ for any $w_2 \in W_2$.

29. We must show $W_1 + W_2 = M_{n\times n}(F)$ and $W_1 \cap W_2 = \{0\}$.

Given $A \in M_{n\times n}(F)$, let $A_u, A_d, A_\ell \in M_{n\times n}(F)$ denote its respective strictly upper triangular, diagonal, and strictly lower triangular parts. To be more precise, their entries are given by

$$(A_u)_{ij} = \begin{cases} A_{ij} & \text{if } i < j \\ 0 & \text{if } i \geq j \end{cases} \quad (A_d)_{ij} = \begin{cases} A_{ij} & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad (A_\ell)_{ij} = \begin{cases} A_{ij} & \text{if } i > j \\ 0 & \text{if } i \leq j \end{cases}$$

Thus we have $A = A_u + A_d + A_\ell$ and $(A^t)_u = (A_\ell)^t, (A^t)_d = A_d, (A^t)_\ell = (A_u)^t$.

Observe that W_1 contains $A_u^t + A_d + A_u$. Observe that W_2 contains A_ℓ and A_u^t, hence also the difference $A_\ell - A_u^t$. Since $A = (A_u^t + A_d + A_u) + (A_\ell - A_u^t)$, we see that $W_1 + W_2 = M_{n\times n}(F)$.

On the other hand, if $A \in W_1 \cap W_2$, then $A = A_\ell, A_d = 0, A_u = 0$, but also $A = A^t$ so $(A_\ell)^t = A_u = 0$. Thus $A = 0$ and so $W_1 \cap W_2 = \{0\}$.

Note: the book's assumption that $char F \neq 2$ is a red herring and is not needed!

30. Suppose $V = W_1 \oplus W_2$. Let $v \in V$. Since $V = W_1 + W_2$, there exist $w_1 \in W_1, w_2 \in W_2$ such that $v = w_1 + w_2$. Suppose there were another $w_1' \in W_1, w_2' \in W_2$ such that $v = w_1' + w_2'$. Then we have $w_1 + w_2 = v = w_1' + w_2'$ and so $w_1 - w_1' = w_2' - w_2$. But since $W_1 \cap W_2 = \{0\}$, we have $w_1 - w_1' = 0 = w_2' - w_2$ and so $w_1 = w_1', w_2 = w_2'$.

Conversely, suppose for each $v \in V$, there exist unique $w_1 \in W_1, w_2 \in W_2$ such that $v = w_1 + w_2$. Then we immediately have $V = W_1 + W_2$. Now suppose $w \in W_1 \cap W_2$. Then we can write $w = w + 0$ or alternatively $w = 0 + w$. Thus we must have $w = 0$ and so $W_1 \cap W_2 = \{0\}$.

Section 1.4.

5. (a) Yes: $(2, -1, 1) = (1, 0, 2) - (-1, 1, 1)$.
(b) No.
(c) No.
(d) Yes: $(2, -1, 1, -3) = 2(1, 0, 1, -1) - (0, 1, 1, 1)$.
(e) Yes: $-x^3 + 2x^2 + 3x + 3 = -1(x^3 + x^2 + x + 1) + 3(x^2 + x + 1) + (x + 1)$.
(f) No.
(g) Yes:

$$\begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix} = 3\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} + 4\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + -2\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

(h) No.

15. Suppose $v \in span(S_1 \cap S_2)$, so $v = a_1 v_1 + \cdots + a_k v_k$, for vectors $v_1, \ldots, v_k \in S_1 \cap S_2$ and scalars $a_1, \ldots, a_k \in F$. Then clearly $v_1, \ldots, v_k \in S_1$ so $v \in span(S_1)$, and similarly, $v_1, \ldots, v_k \in S_2$ so $v \in span(S_2)$. Thus $v \in span(S_1) \cap span(S_2)$.

Example when $span(S_1 \cap S_2) = span(S_1) \cap span(S_2)$: take $V = S_1 = S_2$.

Example when $span(S_1 \cap S_2) \neq span(S_1) \cap span(S_2)$: take $V = \mathbb{R}$, $S_1 = \{1\}$, $S_2 = \{2\}$.

Additional problem: Prove that every field F contains either the field of rational numbers $\mathbb{Q}$ or a finite field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ for a prime p.

Solution: Take the smallest subset of F containing $0, 1$ and closed under addition, multiplication, and forming additive and multiplicative inverses. Some notation used in the following argument to justify the above strategy: Given a positive integer m, we will write $[m] \in F$ for the sum of m copies $1 + \cdots + 1$ of the multiplicative unit $1 \in F$. We will also write $[-m] \in F$ for the additive inverse $-[m]$.

Suppose first that $char F = 0$ so that $[n] \neq 0$ for any $n > 0$. Consider all elements of the form $[m][n]^{-1} \in F$, for integers m, n with $n \neq 0$. From the field axioms, we see that they form a subset closed under addition, multiplication and forming additive and multiplicative inverses. We claim such elements are distinct whenever the corresponding rational numbers are distinct and hence form a copy of $\mathbb{Q}$. From the field axioms, it suffices to see that any $[m][n]^{-1} \in F$ equal to 0 is in fact of the form $[0][n]^{-1}$. But if $[m][n]^{-1} = 0$ then $[m] = [m][n]^{-1}[n] = 0$ so $[m] = 0$ so $m = 0$ since $char F = 0$.

Now if $char F \neq 0$, then we have seen it is equal to some prime p. Consider the p elements of the form $[m] \in F$, for integers m. From the field axioms, we see that they form a field and in fact a copy of $\mathbb{F}_p$.