Solutions to Homework #13.

Section 7.1

1. (a) T (b) F (c) T (d) T (e) F (f) F (g) T (h) T

3a. To find the eigenvalues of T, set $T(f) = \lambda f$, which gives $f' = (2 - \lambda)f$. Thus $2 - \lambda$ must be an eigenvalue of the derivative operator, whose only eigenvalue is 0 (because the derivative reduces the degree of a polynomial by one). So the only eigenvalue of T is $\lambda = 2$, hence the characteristic polynomial is $-(t-2)^3$. Then $T - 2I$ is the operator $f \mapsto -f'$, whose nullspace is one-dimensional, spanned by the constant polynomial 1. $(T - 2I)^2$ is the operator which sends $f \mapsto f''$, whose nullspace is 2-D (spanned by $1, x$), and $(T - 2I)^2$ is the operator $f \mapsto -f'''$, which is the zero map on $P_2(\mathbb{R})$. To find our cycle, pick something which is not in the nullspace of $(T-2I)^2$, say x^2. Then the next vector in the cycle is $-2x$, and the third vector 2. In terms of this basis $\{2, -2x, x^2\}$, the matrix for T is

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

13. By Thm 7.8, $V = K_{\lambda_1} \oplus \cdots \oplus K_{\lambda_k}$ (proof omitted). For each $i = 1, \ldots, k$, choose a basis β_i which is a union of cycles of generalized eigenvectors for $T_{K_{\lambda_i}}$. Then $J_i = [T_{K_{\lambda_i}}]_{\beta_i}$ is a Jordan form for $T_{K_{\lambda_i}}$. But also, setting $\beta = \bigcup_{i=1}^k \beta_i$, we see that β is a basis for V consisting of cycles of generalized eigenvectors for T, so $[T]_\beta$ is a Jordan form for T, and since $\beta = \bigcup_{i=1}^k \beta_i$, $[T]_\beta = J_1 \oplus \cdots \oplus J_k$.

Section 7.2

1. (a) T (b) T F (d) T (e) T (f) F (g) F (h) T

4a. By example 5, we know that $A = QJ_AQ^{-1}$, where $J_A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$. The columns of Q will be cycles from the generalized eigenspaces for A. for $\lambda = 1$, we just have a cycle of length one, i.e., and eigenvector, say $(1, 2, 1)$. For $\lambda = 2$ we have the cycle $(1, 2, 0)$ and $(A - 2I)(1, 2, 0) = (1, 1, -1)$. So we may take Q to be the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & -1 & 0 \end{pmatrix}$$

5a. The matrix of T with respect to this basis is

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

For $\lambda = 1$, we produce the cycle $e_3, 2e_2, 2e_1$. For $\lambda = 2$, we have just the one eigenvector e_4. Converting back to the original basis, we get the Jordan canonical basis $\{2e^t, 2te^t, t^2e^t, e^{2t}\}$, with

Jordan form

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

6. Since the Jordan canonical form is determined by the dimensions of the spaces $N((A-\lambda)^k)$, it will be enough to show that these dimensions agree for A and A^T. But this is clear since $(A-\lambda I)^T = A^T - \lambda I$ and taking the transpose commutes with taking powers, so (using the fact that the ranks of B and B^T are equal, for any square matrix B)

$$\text{rank}(A^T - \lambda I)^r = \text{rank}((A-\lambda I)^T)^r = \text{rank}(A-\lambda I)^r$$

Section 7.3

1. (a) F (b) T (c) F (d) T (e) T (f) F (g) F (h) T (i) T

2d. This matrix has the sole eigenvalue 2, with one-dimensional eigenspace. This implies that its Jordan form consists of a single block, so the minimal polynomial is $(t-2)^3$. This is because, for each eigenvalue λ, the factor $(t-\lambda)$ occurs in the minimal polynomial with multiplicity equal to the length of the largest Jordan block associated to λ (in this case, three).

5. We know that the minimal polynomial $p(t)$ of T must divide the polynomial $t^3-2t^2+1 = t(t-1)^2$. Since T is diagonalizable, $p(t)$ must factor into distinct linear factors. Thus the only possibilities are

1. $p(t) = t$
2. $p(t) = t-1$
3. $p(t) = t(t-1)$

All three possibilites are realized. In fact, in the first case, T must have the sole eigenvalue $\lambda = 0$, hence is similar to the zero map, hence is equal to the zero map (because if $QAQ^{-1} = 0$, then we get $A = 0$). Similarly, in case 2, T is the identity map. Finally, in case 3, T is similar to the map given by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.