Solutions to Homework #11.

Section 6.1

4. (a) We are left to check some of the defining properties of an inner product. Namely,

$$\begin{aligned}\langle cA, B\rangle &= \mathrm{Tr}(B^*cA)\\ &= c\,\mathrm{Tr}(B^*A)\\ &= c\langle A, B\rangle\end{aligned}$$

and

$$\begin{aligned}\overline{\langle B, A\rangle} &= \overline{\mathrm{Tr}(A^*B)}\\ &= \mathrm{Tr}(\overline{A^*B})\\ &= \mathrm{Tr}((\overline{A^*B})^T)\\ &= \mathrm{Tr}(\overline{B^*}(A^*)^*)\\ &= \mathrm{Tr}(B^*A)\\ &= \langle A, B\rangle.\end{aligned}$$

8. (a) In $\langle (a,b),(c,d)\rangle = ac - bd$ set $a = c = 0$ and $b = d = 1$ to get $\langle (0,1),(0,1)\rangle = -1 < 0$ which contradicts the positivity property of inner products.

9. (a) Let the basis be $\beta = \{z_1, \ldots, z_n\}$. We know that $\langle x, z_i\rangle = 0$ for $i = 1, \ldots, n$. Since $\beta$ is a basis, there exists scalars $\alpha_i$ such that $x = \sum \alpha_i z_i$. Then using the properties of an inner product

$$\begin{aligned}\langle x, x\rangle &= \langle x, \sum \alpha_i z_i\rangle\\ &= \sum \overline{\alpha}_i \langle x, z_i\rangle\\ &= 0\end{aligned}$$

Therefore, $x = 0$.

(b) We note that $\langle x - y, z_i\rangle = 0$ and we apply the result from part (a) to get that $x - y = 0$.

11. The equality follows easily from

$$\begin{aligned}\langle x+y, x+y\rangle &= \langle x, x+y\rangle + \langle y, x+y\rangle\\ &= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle\end{aligned}$$

and

$$\langle x-y, x-y\rangle = \langle x, x\rangle + \langle x, -y\rangle + \langle -y, x\rangle + \langle -y, -y\rangle.$$

12. First note that by orthogonality $\langle v_i, \sum_j \alpha_j v_j\rangle = \sum_j \bar{\alpha}_j \langle v_i, v_j\rangle = \bar{\alpha}_i \langle v_i, v_i\rangle$ since $\langle v_i, v_j\rangle = 0$ if $i \neq j$. Using this

$$\begin{aligned}\langle \sum_i a_i v_i, \sum_j \alpha_j v_j\rangle &= \sum_i \alpha_i \langle v_i, \sum_j \alpha_j v_j\rangle\\ &= \sum_i \alpha_i \bar{\alpha}_i \langle v_i, v_i\rangle.\end{aligned}$$

Section 6.2

2. (b) $w_1 = (1,1,1), w_2 = (0,1,1), w_3 = (0,0,1)$. First we need to use the Gram-Schmidt process from Theorem 6.4 to get an orthogonal basis $\{v_1, v_2, v_3\}$. For this we set $v_1 = w_1$ and then compute

$$\begin{aligned} v_2 &= w_2 - \frac{\langle w_2, v_1\rangle}{\langle v_1, v_1\rangle} v_1 \\ v_3 &= w_3 - \frac{\langle w_3, v_2\rangle}{\langle v_2, v_2\rangle} v_2 - \frac{\langle w_3, v_1\rangle}{\langle v_1, v_1\rangle} v_1 \end{aligned}$$

Finally, we normalize the vectors to get an orthonormal basis $(\frac{\sqrt{3}}{3}(1,1,1), \frac{\sqrt{6}}{6}(-2,1,1), \frac{\sqrt{2}}{2}(0,-1,1))$. Then the fourier coefficients will be $\frac{2\sqrt{3}}{3}, -\frac{\sqrt{6}}{6}, \frac{\sqrt{2}}{2}$.

6. By Theorem 6.6 there exists $u \in W, y \in W^\perp$ such that $x = u + y$. Note that $y \neq 0$ since we know $x \notin W$. Thus,

$$\begin{aligned} \langle x, y\rangle &= \langle u + y, y\rangle \\ &= \langle u, y\rangle + \langle y, y\rangle \\ &= \langle y, y\rangle \\ &> 0. \end{aligned}$$

13. (a) Pick $u \in S^\perp$. By the definition of the orthogonal complement we have $\langle u, s\rangle = 0\ \forall s \in S$. In particular, since $S_0 \subset S$, we have $\langle u, s\rangle = 0\ \forall s \in S_0$. This means that $u \in S_0^\perp$.

(b) Let $u \in S$. Then for any $s \in S^\perp$ we have $\langle u, s\rangle = 0$. But this is exactly what it means to be in the orthogonal complement of $S^\perp$. Thus, $s \in (S^\perp)^\perp$.

(c) By (b) $W \subset (W^\perp)^\perp$. We need to prove equality. Suppose that $W \neq (W^\perp)^\perp$. Then there exists $x \in (W^\perp)^\perp$ such that $x \notin W$. By Exercise 6 there exists $y \in V$ such that $y \in W^\perp$ and $\langle x, y\rangle \neq 0$. But this contradicts the fact that $x \in (W^\perp)^\perp$.

(d) By Theorem 6.6 we have $v = W + W^\perp$. We just need to show $W \cap W^\perp = \{0\}$. Say $w \in W \cap W^\perp$. Then $\langle w, w\rangle = 0$ since we have the inner product of something from $W$ with something from $W^\perp$. This implies that $w = 0$.

12. First note that if $y = \alpha, z = \beta$ are free variables then we can write that $x = -3\alpha + 2\beta$. This gives us the basis $(-3,1,0), (2,0,1)$ for $W$. Now we use Gram-Schmidt on this basis to get $v_1 = \frac{1}{\sqrt{10}}(-310)$ and $v_2 = \frac{5}{7}(1/5, 3/5, 1)$. Therefore, the projection of $u$ on $W$ will be

$$\langle u, v_1\rangle v_1 + \langle u, v_2\rangle v_2 = \frac{1}{14}(29, 17, 40).$$