Solutions to Homework #11.

Section 6.1

4. (a) We are left to check some of the defining properties of an inner product. Namely,

\[ \langle cA, B \rangle = \text{Tr}(B^*cA) = c\text{Tr}(B^*A) = c\langle A, B \rangle \]

and

\[ \langle B, A \rangle = \text{Tr}(A^*B) = \text{Tr}(A^*B) = \text{Tr}((A^*B)^T) = \text{Tr}(B^*(A^*)^*) = \text{Tr}(B^*A) = \langle A, B \rangle. \]

8. (a) In \( \langle (a, b), (c, d) \rangle = ac - bd \) set \( a = c = 0 \) and \( b = d = 1 \) to get \( \langle (0, 1), (0, 1) \rangle = -1 < 0 \) which contradicts the positivity property of inner products.

9. (a) Let the basis be \( \beta = \{z_1, \ldots, z_n\} \). We know that \( \langle x, z_i \rangle = 0 \) for \( i = 1, \ldots, n \). Since \( \beta \) is a basis, there exists scalars \( \alpha_i \) such that \( x = \sum \alpha_i z_i \). Then using the properties of an inner product

\[ \langle x, x \rangle = \langle x, \sum \alpha_i z_i \rangle = \sum \alpha_i \langle x, z_i \rangle = 0 \]

Therefore, \( x = 0 \).

(b) We note that \( \langle x - y, z_i \rangle = 0 \) and we apply the result from part (a) to get that \( x - y = 0 \).

11. The equality follows easily from

\[ \langle x + y, x + y \rangle = \langle x, x + y \rangle + \langle y, x + y \rangle = \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle \]

and

\[ \langle x - y, x - y \rangle = \langle x, x \rangle + \langle x, -y \rangle + \langle -y, x \rangle + \langle -y, -y \rangle. \]

12. First note that by orthogonality \( \langle v_i, \sum_j \alpha_j v_j \rangle = \sum_j \alpha_j \langle v_i, v_j \rangle = \tilde{\alpha}_i \langle v_i, v_i \rangle \) since \( \langle v_i, v_j \rangle = 0 \) if \( i \neq j \). Using this

\[ \langle \sum_i \alpha_i v_i, \sum_j \alpha_j v_j \rangle = \sum_i \alpha_i \langle v_i, \sum_j \alpha_j v_j \rangle = \sum_i \alpha_i \tilde{\alpha}_i \langle v_i, v_i \rangle. \]
2. (b) \( w_1 = (1, 1, 1), w_2 = (0, 1, 1), w_3 = (0, 0, 1) \). First we need to use the Gram-Schmidt process from Theorem 6.4 to get an orthogonal basis \{v_1, v_2, v_3\}. For this we set \( v_1 = w_1 \) and then compute

\[
\begin{align*}
v_2 &= w_2 - \frac{\langle w_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 \\
v_3 &= w_3 - \frac{\langle w_3, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2 - \frac{\langle w_3, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1
\end{align*}
\]

Finally, we normalize the vectors to get an orthonormal basis \((\sqrt{3}/3, 1, 1/\sqrt{6}, -2, 1, 1, \sqrt{2}/2, 0, -1, 1)\).

Then the Fourier coefficients will be \(2\sqrt{3}/3, -\sqrt{6}/6, \sqrt{2}/2\).

6. By Theorem 6.6 there exists \( u \in W, y \in W^\perp \) such that \( x = u + y \). Note that \( y \neq 0 \) since we know \( x \notin W \). Thus,

\[
\langle x, y \rangle = \langle u + y, y \rangle = \langle u, y \rangle + \langle y, y \rangle = \langle y, y \rangle > 0.
\]

13. (a) Pick \( u \in S^\perp \). By the definition of the orthogonal complement we have \( \langle u, s \rangle = 0 \) \( \forall s \in S \). In particular, since \( S_0 \subset S \), we have \( \langle u, s \rangle = 0 \) \( \forall s \in S_0 \). This means that \( u \in S_0^\perp \).

(b) Let \( u \in S \). Then for any \( s \in S^\perp \) we have \( \langle u, s \rangle = 0 \). But this is exactly what it means to be in the orthogonal complement of \( S^\perp \). Thus, \( s \in (S^\perp)^\perp \).

(c) By (b) \( W \subset (W^\perp)^\perp \). We need to prove equality. Suppose that \( W \neq (W^\perp)^\perp \). Then there exists \( x \in (W^\perp)^\perp \) such that \( x \notin W \). By Exercise 6 there exists \( y \in V \) such that \( y \in W^\perp \) and \( \langle x, y \rangle \neq 0 \). But this contradicts the fact that \( x \in (W^\perp)^\perp \).

(d) By Theorem 6.6 we have \( v = W + W^\perp \). We just need to show \( W \cap W^\perp = \{0\} \). Say \( w \in W \cap W^\perp \). Then \( \langle w, w \rangle = 0 \) since we have the inner product of something from \( W \) with something from \( W^\perp \). This implies that \( w = 0 \).

12. First note that if \( y = \alpha, z = \beta \) are free variables then we can write that \( x = -3\alpha + 2\beta \). This gives us the basis \((-3, 1, 0), (2, 0, 1)\) for \( W \). Now we use Gram-Schmidt on this basis to get \( v_1 = \frac{1}{\sqrt{10}}(-310) \) and \( v_2 = \frac{5}{7}(1/5, 3/5, 1) \). Therefore, the projection of \( u \) on \( W \) will be

\[
\langle u, v_1 \rangle v_1 + \langle u, v_2 \rangle v_2 = \frac{1}{14}(29, 17, 40).
\]