SUBCRITICAL SCATTERING FOR DEFOCUSING NLS

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ABSTRACT. We survey some known results concerning the asymptotic behavior of solutions to defocusing nonlinear Schrödinger equations. In particular, we discuss the H^1 scattering theory for intercritical NLS, as well as the scattering theory in weighted spaces for the mass-subcritical case. We also discuss an instance of modified scattering in the long-range case.

CONTENTS

1. INTRODUCTION

In this note, we survey some known results concerning the asymptotic behavior of solutions to nonlinear Schrödinger equations (NLS). In particular, we study the initial-value problem for power-type equations of the form

$$\begin{cases} (i\partial_t + \Delta)u = \mu|u|^p u, \\ u(0) = u_0. \end{cases} \tag{1.1}$$

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Here $u : \mathbb{R}_t \times \mathbb{R}^d_x \to \mathbb{C}$ is a complex-valued function of space-time with $d \geq 1$. The coefficient $\mu \in \{\pm 1\}$ corresponds to the defocusing and focusing cases, respectively; we will be concerned primarily with the defocusing case. Restrictions on the power of the nonlinearity $p > 0$ and conditions on the initial data u_0 will be discussed below.

The equation (1.1) enjoys several symmetries and conservation laws. The following non-exhaustive list, which we will return to in Section 5, will be relevant in the sequel.

- The time translation symmetry $u(t,x) \mapsto u(t+t_0,x)$ for $t_0 \in \mathbb{R}$ corresponds to the conservation of the *energy* (or *Hamiltonian*), defined by the sum of the kinetic and potential energy:
$$\mathcal{E}[u(t)] := \int_{\mathbb{R}^d} \tfrac{1}{2}|\nabla u(t,x)|^2 + \tfrac{\mu}{p+2}|u(t,x)|^{p+2}\,dx. \tag{1.2}$$
- The space translation symmetry $u(t,x) \mapsto u(t,x+x_0)$ for $x_0 \in \mathbb{R}^d$ corresponds to the conservation of the *momentum*, defined by
$$\mathcal{P}[u(t)] := 2\,\mathrm{Im}\int_{\mathbb{R}^d} \bar{u}(t,x)\nabla u(t,x)\,dx. \tag{1.3}$$
- The gauge symmetry $u(t,x) \mapsto e^{i\theta}u(t,x)$ for $\theta \in \mathbb{R}$ corresponds to the conservation of the *mass*, defined by
$$\mathcal{M}[u(t)] := \int_{\mathbb{R}^d} |u(t,x)|^2\,dx. \tag{1.4}$$
- The *scaling symmetry*
$$u(t,x) \mapsto u^\lambda(t,x) := \lambda^{\frac{2}{p}} u(\lambda^2 t, \lambda x) \quad \text{for} \quad \lambda > 0 \tag{1.5}$$
leads to a notion of *criticality* for (1.1) in the following sense. If one selects initial data u_0 from a homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^d)$ for some $s \in \mathbb{R}$, then
$$\|u_0^\lambda\|_{\dot{H}^s(\mathbb{R}^d)} = \lambda^{s-(\frac{d}{2}-\frac{2}{p})}\|u_0\|_{\dot{H}^s(\mathbb{R}^d)}.$$
Defining the *critical regularity* for (1.1) by
$$s_c = \tfrac{d}{2} - \tfrac{2}{p}, \tag{1.6}$$
one verifies that the $\dot{H}^{s_c}(\mathbb{R}^d)$-norm is invariant under scaling. Choosing $s = s_c$ gives the *critical* initial-value problem for (1.1), while choosing $s > s_c$ or $s < s_c$ gives *subcritical* and *supercritical* problems, respectively.

Special cases of (1.1) arise when one of the conserved quantities is invariant under the scaling (1.5).

- The *mass-critical* case is given by $p = \frac{4}{d}$. In this case, one has $s_c = 0$ and $\mathcal{M}[u^\lambda] \equiv \mathcal{M}[u]$.
- The *energy-critical* case is given by $p = \frac{4}{d-2}$ in dimensions $d \geq 3$. In this case, one has $s_c = 1$ and $\mathcal{E}[u^\lambda] \equiv \mathcal{E}[u]$.

We typically do not speak of the 'momentum-critical' case, which would correspond to $p = \frac{4}{d-1}$, due to the fact that the momentum is not a coercive quantity. Nonetheless, as we will see below, identities related to the conservation of momentum play an important role in the analysis of solutions to the nonlinear equation.

The critical initial-value problem for (1.1), especially in the mass- and energy-critical cases, has been the catalyst for a great deal of mathematical development in the field of dispersive equations. Such problems are well beyond the scope of this note. We will consider only subcritical problems in the *energy-subcritical* regime (i.e. $s_c < 1$), which greatly simplifies the well-posedness theory (see Section 4). We will also primarily consider the defocusing case, which corresponds to choosing $\mu = 1$ in (1.1) and guarantees that the energy (1.2) controls both of its constituent pieces. In particular, we will only consider problems for which we are guaranteed to have global (in time) solutions (see Theorem 4.2); our primary interest will then be to study the behavior of solutions as $t \to +\infty$ (say).

Specifically, we will consider the question of *scattering*: given a global solution $u(t)$ to (1.1), does there exist a solution $v(t)$ to the linear Schrödinger equation

$$(i\partial_t + \Delta)v = 0$$

such that

$$\lim_{t\to\infty} \|u(t) - v(t)\| = 0$$

in a suitable norm? Scattering is essentially the simplest possible long-time behavior: the nonlinear effects simply become negligible as $t \to \infty$.

As we will see, scattering will typically follow from appropriate decay estimates for solutions to the nonlinear equation. This may refer either to pointwise-in-time decay estimates for L_x^r-norms of the solution, or to global space-time bounds in mixed Lebesgue spaces of the form $L_t^q L_x^r$ (see Section 2 for this notation). In either case, the heuristic is as follows: if the solution ever becomes small, then the nonlinearity $|u|^p u$ will be even smaller. In particular, one may be able to get so much control over the nonlinearity that it can be shown to become negligible. However, as we will see, this heuristic may break down if the power p becomes too small.

The primary tools available for establishing decay to nonlinear solutions are a collection of estimates known as virial or Morawetz estimates, which follow from identities related to the conservation of momentum. It is in proving these estimates that the defocusing nature of the nonlinearity plays the largest role. We discuss Morawetz and virial estimates in Section 5.

We now briefly describe the main results we will cover in this note.

1.1. **Intercritical NLS.**

We first consider the defocusing *intercritical* NLS in dimensions $d \geq 3$; this corresponds to critical regularities $s_c \in (0,1)$, or equivalently

$$\tfrac{4}{d} < p < \tfrac{4}{d-2}. \tag{1.7}$$

We consider initial data $u_0 \in H^1(\mathbb{R}^d)$, which guarantees the existence of a unique, global solution $u(t)$ with finite mass and energy (see Section 4).

We present a result due originally to Ginibre and Velo, namely, that scattering holds in H^1 (see Theorem 6.4). We present a proof similar to that of Tao, Visan, and Zhang, which relies on the *interaction Morawetz estimate* (see Section 5).

1.2. **Mass-subcritical NLS.**

We next study the *mass-subcritical* NLS, which corresponds to critical regularities $s_c < 0$, or equivalently

$$0 < p < \tfrac{4}{d}. \tag{1.8}$$

In this case, initial data $u_0 \in L^2(\mathbb{R}^d)$ leads to a unique, global solution $u(t)$ of finite mass, even in the focusing case (cf. Section 4). To study the question of scattering, however, it is natural to restrict to the defocusing problem and to prescribe initial data in the weighted Sobolev space Σ, defined by the norm

$$\|u\|_{\Sigma}^2 := \|u\|_{L^2(\mathbb{R}^d)}^2 + \|\nabla u\|_{L^2(\mathbb{R}^d)}^2 + \|xu\|_{L^2(\mathbb{R}^d)}^2. \tag{1.9}$$

In particular, these assumptions suffice to access the so-called *pseudoconformal energy estimate* (see Section 5).

- We will first prove a result of Tsutsumi and Yajima, which states that for $\frac{2}{d} < p < \frac{4}{d}$, scattering holds in L^2 (see Theorem 7.3). This is called the *short-range* case.
- We will next prove a result of Cazenave and Weissler, which states that scattering holds in a stronger topology for a restricted range of p in the short-range case (see Theorem 7.5).
- We will then prove a result of Strauss and Barab, which states that for $0 < p \leq \frac{2}{d}$, scattering cannot hold in L^2 unless the solution is identically zero (see Theorem 7.7). This is called the *long-range* case.

1.3. **Modified scattering.** Finally, we will consider the borderline case $p = \frac{2}{d}$. In light of Theorem 7.7, scattering does not hold; however, in this case it is possible to identify a suitable correction to linear scattering and to prove a form of 'modified' scattering.

Currently, results are only available in low dimensions $d \in \{1, 2, 3\}$ and for small solutions in suitable weighted Sobolev spaces. The exception to this is dimension $d = 1$, in which case the equation is completely integrable; one can then establish a large data result via inverse scattering techniques.

For technical simplicity, we will consider the problem in dimension $d = 1$ for small data in Σ. At least three proofs exist to treat this case (due to Hayashi and Naumkin, Kato and Pusateri, and Ifrim and Tataru), but all have a similar flavor: one proceeds by a bootstrap argument, where the estimates only close if one incorporates an appropriate correction term. See Theorem 8.1.

1.4. **Outline of the paper.**

- In Section 2, we set up notation to be used throughout the rest of the paper.
- In Section 3, we collect some useful information about the underlying linear Schrödinger equation.
- In Section 4, we discuss the well-posedness theory for (1.1).
- In Section 5, we discuss conservation laws and their relatives, the Morawetz and virial identites.
- In Section 6, we consider the defocusing intercritical problem.
- In Section 7, we consider the mass-subcritical problem.
- In Section 8, we discuss modified scattering.

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Note to the reader. For the time being, I will consider this note to be a work in progress. Currently the most glaring omission is a complete lack of references. I hope to continue to add more detail, more results, and references as time goes on. Please feel free to suggest corrections, references, and additional material to present.

2. Notation

We write $A \lesssim B$ or $B \gtrsim A$ to denote $A \leq CB$ for some $C > 0$ that may depend on the dimension, the power p of the nonlinearity, or implicit constants in various functional inequalities. If $A \lesssim B \lesssim A$, we write $A \sim B$.

For a function $f : \mathbb{R}^d \to \mathbb{C}$, we write $\|f\|_{L_x^r(\mathbb{R}^d)}$ or $\|f\|_{L_x^r}$ for the L^r-norm of f, $1 \leq r \leq \infty$. For a function $u : I \times \mathbb{R}^d \to \mathbb{C}$ for some interval $I \subset \mathbb{R}$, we write

$$\|u\|_{L_t^q L_x^r(I\times\mathbb{R}^d)} = \big\| \|u(t)\|_{L_x^r(\mathbb{R}^d)} \big\|_{L_t^q(I)}$$

where $1 \leq q, r \leq \infty$.

For $r \in [1, \infty]$, we let $r' \in [1, \infty]$ denote the Hölder dual of r, that is, the solution to $\frac{1}{r} + \frac{1}{r'} = 1$.

We denote by $\mathcal{F}u = \widehat{u}$ the Fourier transform of a function u, defined by

$$\mathcal{F}u(\xi) = \widehat{u}(\xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} u(x)\, dx,$$

and we let $\mathcal{F}^{-1}f = \check{f}$ denote the inverse Fourier transform. For a given function $m : \mathbb{R}^d \to \mathbb{R}$, we may define the Fourier multiplier operator $m(i\nabla) = \mathcal{F}^{-1}m(\xi)\mathcal{F}$. In particular, $m(i\nabla)f = \check{m} * f$.

Special cases include the fractional derivatives $|\nabla|^s$ corresponding to $m(\xi) = |\xi|^s$ for $s \in \mathbb{R}$, along with the free Schrödinger propagator $e^{it\Delta}$ corresponding to $m(\xi) = e^{-it|\xi|^2}$ (see Section 3). We also define $\langle\nabla\rangle^s$ to be the Fourier multiplier operator with symbol $m(\xi) = (1 + |\xi|^2)^{\frac{s}{2}}$.

These derivative operators define the homogeneous and inhomogeneous Sobolev spaces $\dot{H}^s$ and H^s via the norms

$$\|f\|_{\dot{H}^s(\mathbb{R}^d)} = \||\nabla|^s f\|_{L_x^2(\mathbb{R}^d)}, \quad \|f\|_{H^s(\mathbb{R}^d)} = \|\langle\nabla\rangle^s f\|_{L_x^2(\mathbb{R}^d)}.$$

3. The linear Schrödinger equation

Solutions to the linear Schrödinger equation

$$\begin{cases} (i\partial_t + \Delta)v = 0, \\ v(0) = \phi \end{cases} \tag{3.1}$$

are given by $v(t) = e^{it\Delta}\phi$, where $e^{it\Delta} = \mathcal{F}^{-1}e^{-it|\xi|^2}\mathcal{F}$ is the linear Schrödinger propagator. More generally, variation of parameters implies that the solution to the inhomogeneous Schrödinger equation

$$\begin{cases} (i\partial_t + \Delta)v = F, \\ v(0) = \phi \end{cases} \tag{3.2}$$

is given by

$$v(t) = e^{it\Delta}\phi - i\int_0^t e^{i(t-s)\Delta}F(s)\, ds.$$

Using the definition of $e^{it\Delta}$ and Plancherel's theorem, it is clear that

$$\|e^{it\Delta}\phi\|_{L_x^2(\mathbb{R}^d)} \equiv \|\phi\|_{L_x^2(\mathbb{R}^d)}. \tag{3.3}$$

In physical space, one can derive the formula

$$[e^{it\Delta}\phi](x) = (4\pi it)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{i|x-y|^2/4t}\phi(y)\, dy \quad \text{for all} \quad t \neq 0. \tag{3.4}$$

From this identity, one can read off the *dispersive estimate*

$$\|e^{it\Delta}\phi\|_{L_x^\infty(\mathbb{R}^d)} \lesssim |t|^{-\frac{d}{2}}\|\phi\|_{L_x^1(\mathbb{R}^d)} \quad \text{for all} \quad t \neq 0. \tag{3.5}$$

Interpolating with (3.3) yields a more general class of dispersive estimates, namely

$$\|e^{it\Delta}\phi\|_{L_x^r(\mathbb{R}^d)} \lesssim |t|^{-(\frac{d}{2}-\frac{d}{r})}\|\phi\|_{L_x^{r'}(\mathbb{R}^d)} \quad \text{for all} \quad t \neq 0, \tag{3.6}$$

where $2 \leq r \leq \infty$.

These estimates may also be used to establish global space-time bounds for solutions to the linear Schrödinger equation, as well as for solutions to inhomogeneous Schrödinger equations. Such estimates are known as Strichartz estimates; they play a key role in the well-posedness theory for (1.1).

To properly state the estimates, we introduce the following terminology: we call a pair of exponents (q, r) *admissible* if

$$2 \leq q, r \leq \infty, \quad \tfrac{2}{q} + \tfrac{d}{r} = \tfrac{d}{2}, \quad (d, q, r) \neq (2, 2, \infty).$$

We call a pair (α, β) *dual admissible* if (α', β') is admissible.

Theorem 3.1 (Strichartz estimates). *Let (q, r) be an admissible pair and (α, β) a dual admissible pair. Then*

$$\|e^{it\Delta}\phi\|_{L_t^q L_x^r(I\times\mathbb{R}^d)} \lesssim \|\phi\|_{L_x^2(\mathbb{R}^d)},$$

$$\left\|\int_0^t e^{i(t-s)\Delta}F(s)\,ds\right\|_{L_t^q L_x^r(I\times\mathbb{R}^d)} \lesssim \|F\|_{L_t^\alpha L_x^\beta(I\times\mathbb{R}^d)}$$

for any interval $I \subset \mathbb{R}$.

From (3.4), we can also read off a very useful factorization of $e^{it\Delta}$, namely,

$$e^{it\Delta} = M(t)D(t)\mathcal{F}M(t), \tag{3.7}$$

where

$$M(t) = e^{i|x|^2/4t} \quad \text{and} \quad [D(t)f] = (2it)^{-\frac{d}{2}}f(\tfrac{x}{2t}).$$

In particular, (3.7) and the dominated convergence theorem imply the following lemma, which describes the asymptotics of solutions to (3.1):

Lemma 3.2 (Fraunhofer formula). *For any $\phi \in L_x^2(\mathbb{R}^d)$,*

$$\lim_{t\to\infty}\|e^{it\Delta}\phi - M(t)D(t)\mathcal{F}\phi\|_{L_x^2} = 0.$$

Finally, we introduce the operator

$$J(t) := x + 2it\nabla, \tag{3.8}$$

which will play an important role in the scattering theory in weighted spaces. By direct computation and (3.7), we have

$$J(t) = M(t)2it\nabla M(-t) = e^{it\Delta}xe^{-it\Delta}. \tag{3.9}$$

In particular,

$$J(t)e^{it\Delta}\phi = e^{it\Delta}x\phi,$$

which suggests some physical interpretation for $J(t)$, namely, it measures how the center of mass evolves for linear solutions.

Control over $J(t)$ in L_x^2 implies decay as $|t| \to \infty$; we leave this fact as an exercise to the reader.

Exercise. Let $d \geq 3$. Show that

$$\|f\|_{L_x^{\frac{2d}{d-2}}(\mathbb{R}^d)} \lesssim |t|^{-1} \|J(t)f\|_{L_x^2(\mathbb{R}^d)}.$$

4. Well-posedness

In this section, we briefly discuss the basic questions of existence and uniqueness of solutions to (1.1). As discussed in the introduction, we will restrict our discussion to energy-subcritical nonlinearities (i.e. $s_c < 1$) and we will select initial data from the space $H^1(\mathbb{R}^d)$. This makes (1.1) a subcritical problem and results in a simple well-posedness theory. Much more detail can be found, for example, in the book of Cazenave.

By a *solution* to (1.1) on an interval $I \ni 0$, we mean a function $u : I \times \mathbb{R}^d$ satisfying the Duhamel formula

$$u(t) = e^{it\Delta} u_0 - i\mu \int_0^t e^{i(t-s)\Delta} (|u|^p u)(s)\, ds, \quad t \in I, \tag{4.1}$$

such that $u \in C_t H_x^1(K \times \mathbb{R}^d)$ and $\langle \nabla \rangle u \in L_t^q L_x^r(K \times \mathbb{R}^d)$ for all admissible (q, r) and all compact $K \subset I$. We call u global if $I = \mathbb{R}$.

Theorem 4.1 (Local well-posedness)**.** *Let $p > 0$ satisfy*

$$\begin{cases} p < \frac{4}{d-2} & d \geq 3, \\ p < \infty & d \in \{1, 2\}, \end{cases} \tag{4.2}$$

so that in particular $s_c < 1$. For any $u_0 \in H_x^1(\mathbb{R}^d)$, there exists a unique solution u to (1.1) *on some interval $I \ni 0$, where the length of I depends only on $\|u_0\|_{H_x^1(\mathbb{R}^d)}$.*

The proof relies on a fixed point argument using Strichartz estimates, treating the nonlinearity as a perturbation of the linear equation. In particular, one can prove the desired estimates by choosing the time interval sufficiently small. In the subcritical case, the length of the interval depends only on the norm of the initial data; this is in contrast to the critical case (i.e. $u_0 \in \dot{H}^{s_c}$), where the length of the interval actually depends on the profile of the initial data.

For $u_0 \in H_x^1$ and $s_c < 1$, solutions have finite mass and energy. Indeed, in this range the potential energy is controlled via the Gagliardo–Nirenberg inequality by the mass and the kinetic energy. In the defocusing case, the conservation of mass and energy then implies that the solution $u(t)$ remains uniformly bounded in H_x^1 throughout its existence. In particular, one can iterate the subcritical local existence result to deduce the following global-in-time result.

Theorem 4.2 (Global well-posedness in the defocusing case)**.** *Let $p > 0$ satisfy* (4.2) *and $\mu = 1$. For any $u_0 \in H_x^1(\mathbb{R}^d)$, there exists a unique global solution u to* (1.1)*. Furthermore, $u(t)$ remains uniformly bounded in H_x^1.*

Remark 4.3. In the mass-subcritical case $p < \frac{4}{d}$, the length of the local interval of existence depends only on $\|u_0\|_{L_x^2}$. In particular, by the conservation of mass, initial data in L_x^2 lead to global solutions even in the focusing case. In contrast, solutions to (1.1) in the focusing case with $s_c \in (0, 1)$ may blow up in finite time (see the exercise after Lemma 5.3).

Remark 4.4. If one chooses $u_0 \in \Sigma$ (see (1.9)), then the corresponding solution u belongs to $C_t\Sigma$. One can prove this by commuting the vector field J (cf. (3.8)) with the equation and applying standard persistence of regularity arguments. Note that one should not expect $xu(t)$ to remain bounded in L_x^2 (as we expect the solution to spread out); however, one can sometimes prove that $J(t)u(t)$ remains bounded in L_x^2.

5. Conservation laws and Morawetz/virial identities

In this section, we give (formal) proofs of the conservation laws mentioned in Section 1. We then discuss the related Morawetz and virial identities, and deduce some estimates for solutions to the defocusing equation.

Throughout the section, we assume that the solutions under consideration are smooth and decaying enough to justify all the formal computations we carry out. Such assumptions may be removed by standard limiting arguments. See, for instance, the book of Cazenave.

Throughout this section, subscripts denote derivatives and repeated indices are summed. Thus $\Delta u = u_{jj}$, $|\nabla u|^2 = u_k \bar{u}_k$, and so on.

Lemma 5.1 (Conservation laws)**.** *The energy, momentum, and mass of solutions to* (1.1) *defined in* (1.2)–(1.4) *are conserved in time.*

Proof. We compute using (1.1). First,

$$\partial_t[\tfrac{1}{2}|\nabla u|^2 + \tfrac{\mu}{p+2}|u|^{p+2}] = \partial_j \operatorname{Im}[-u_{jk}\bar{u}_k + \mu|u|^p u\bar{u}_j],$$

which implies the conservation of energy. Next,

$$\partial_t[2\operatorname{Im}\bar{u}u_k] = \tfrac{-2p\mu}{p+2}\partial_k|u|^{p+2} + \partial_{jjk}|u|^2 - 4\operatorname{Re}\partial_j(\bar{u}_j u_k), \tag{5.1}$$

which implies the conservation of momentum. Finally,

$$\partial_t|u|^2 = -2\partial_k \operatorname{Im}(\bar{u}u_k), \tag{5.2}$$

which implies the conservation of mass. □

We now turn to a discussion of Morawetz/virial identities. The idea is to pair the momentum with a well-chosen vector field (typically of the form ∇a for some weight function a) and to attempt to demonstrate some monotonicity in time. We begin by considering the classical picture and making a few motivating computations.

Example 5.1. Consider the following simple model for a particle in $\mathbb{R}^d$ under the influence of a potential $V : \mathbb{R}^d \to \mathbb{R}$:

$$\begin{cases} \dot{x} = p, \\ \dot{p} = -\nabla V(x), \end{cases} \tag{5.3}$$

where $\dot{}$ denotes $\frac{d}{dt}$ and $x, p : \mathbb{R} \to \mathbb{R}^d$. Suppose that the potential is repulsive, in the sense that

$$\nabla V(q) \cdot q \leq 0 \quad \text{for all} \quad q \in \mathbb{R}^d.$$

Simple computations then show

$$\tfrac{d}{dt}[p \cdot x] = |p|^2 - x \cdot \nabla V(x) \geq 0,$$

and

$$\tfrac{d}{dt}[p \cdot \tfrac{x}{|x|}] = \tfrac{1}{|x|}\big[|p|^2 - (p \cdot \tfrac{x}{|x|})^2\big] - x \cdot \nabla V(x) \geq 0.$$

In the case of (1.1), we will also be able to demonstrate monotonicity when we pair the momentum with the vector fields x and $\frac{x}{|x|}$. We begin by proving a Morawetz identity with a general weight function.

Lemma 5.2 (Morawetz identity)**.** *Let $a : \mathbb{R}^d \to \mathbb{R}$ and let $u : I \times \mathbb{R}^d \to \mathbb{C}$ be a solution to* (1.1)*. Define the Morawetz quantity*

$$M_a^0(t) = 2\,\mathrm{Im} \int \bar{u}\nabla u \cdot \nabla a \, dx. \tag{5.4}$$

Then

$$M_a^0(t) = \tfrac{d}{dt} \int |u|^2 a \, dx \tag{5.5}$$

and

$$\tfrac{d}{dt} M_a^0(t) = \int \tfrac{2\mu p}{p+2} |u|^{p+2} \Delta a + |u|^2(-\Delta\Delta a) + 4\,\mathrm{Re}\, a_{jk} \bar{u}_j u_k \, dx. \tag{5.6}$$

Proof. First note (5.5) follows from (5.2). Next, (5.6) follows from (5.1) and integration by parts. □

For specific choices of the weight a, (5.6) implies some useful identities. The first, known as the virial identity, results from taking $a(x) = |x|^2$.

Lemma 5.3 (Virial identity)**.** *Let $u : I \times \mathbb{R}^d \to \mathbb{C}$ solve* (1.1)*. Then*

$$\tfrac{d^2}{dt^2} \int |x|^2 |u|^2 \, dx = \tfrac{d}{dt} 2\,\mathrm{Im} \int \bar{u}\nabla u \cdot (2x) \, dx = \int \tfrac{4\mu d p}{p+2} |u|^{p+2} + 8|\nabla u|^2 \, dx. \tag{5.7}$$

Proof. This follows from (5.5) and (5.6), using the weight $a(x) = |x|^2$. Indeed, in this case we have

$$\nabla a = 2x, \quad a_{jk} = 2\delta_{jk}, \quad \Delta a = 2d, \quad \Delta\Delta a = 0,$$

from which the identity follows. □

Exercise. Suppose $\mu = -1$ and $s_c \in [0, 1]$. Suppose $u_0 \in \Sigma$ satisfies $\mathcal{E}[u_0] < 0$. Use the virial identity to prove that the solution to (1.1) with initial data u_0 blows up in finite time in both time directions.

5.1. **Pseudoconformal energy estimate.** We can also use the virial identity to derive the so-called *pseudoconformal energy estimate*, which is related to controlling the L^2-norm of the quantity $J(t)u(t) = (x + 2it\nabla)u(t)$, which was introduced in Section 3. This estimate will play an important role in the scattering theory for the mass-subcritical NLS, and thus we restrict attention to the case $p < \frac{4}{d}$.

Lemma 5.4 (Pseudoconformal energy estimate)**.** *Suppose $\mu = 1$, $p < \frac{4}{d}$, and $u_0 \in \Sigma$. Let $u \in C_t\Sigma(\mathbb{R} \times \mathbb{R}^d)$ be the unique, global solution to* (1.1) *with* $u(0) = u_0$ *given in Section 4. Then*

$$\|J(t)u(t)\|_{L_x^2} + t^2 \|u(t)\|_{L_x^{p+2}}^{p+2} \lesssim t^{2 - \frac{dp}{2}} \quad \text{for all} \quad t \geq 1,$$

where the implicit constant depends on $\|u(1)\|_\Sigma$.

Remark 5.5. Note that this estimate does not prove that $J(t)u(t)$ remains bounded in L_x^2. It does, however, give a decay rate for the potential energy of the solution that matches the rate for linear solutions (cf. (3.6)).

Proof. We first recall $J = x + 2it\nabla$ and write

$$\int |Ju|^2\,dx = \int |x|^2|u|^2 - 2t\,\mathrm{Im}(\bar u\nabla u\cdot 2x) + 4t^2|\nabla u|^2\,dx.$$

By the virial identity (5.7),

$$\tfrac{d}{dt}\int |x|^2|u|^2 - 2t\,\mathrm{Im}(\bar u\nabla u\cdot 2x)\,dx = -\int \tfrac{4\mu dpt}{p+2}|u|^{p+2} + 8t|\nabla u|^2.$$

Thus, noting that conservation of energy gives

$$4t^2\tfrac{d}{dt}\int |\nabla u|^2 = -8t^2\tfrac{d}{dt}\int \tfrac{\mu}{p+2}|u|^{p+2}\,dx,$$

we deduce

$$\tfrac{d}{dt}\int |Ju|^2\,dx = -\int \tfrac{4\mu dpt}{p+2}|u|^{p+2}\,dx - 8t^2\tfrac{d}{dt}\int \tfrac{\mu}{p+2}|u|^{p+2}\,dx.$$

Hence, if we define

$$e(t) := \int |Ju|^2 + \tfrac{8\mu t^2}{p+2}|u|^{p+2}\,dx,$$

then

$$e'(t) = \tfrac{4\mu t(4-dp)}{p+2}\int |u|^{p+2}\,dx = \tfrac{2-\frac{dp}{2}}{t}\,\tfrac{8\mu t^2}{p+2}\int |u|^{p+2}\,dx. \tag{5.8}$$

Now note that Gronwall's inequality implies

$$t^2\|u(t)\|_{L_x^{p+2}}^{p+2} \lesssim t^{2-\frac{dp}{2}},$$

where the implicit constant depends on $e(1)\lesssim \|u(1)\|_\Sigma$. Inserting this estimate back into (5.8) yields

$$e(t)\lesssim t^{2-\frac{dp}{2}},$$

which implies the result. □

5.2. **The Lin–Strauss Morawetz estimate.** Next, we prove an estimate (known as the Lin–Strauss Morawetz estimate, the radial Morawetz estimate, or the classical Morawetz estimate) that results from using Lemma 5.2 with the weight $a(x) = |x|$.

Lemma 5.6 (Lin–Strauss Morawetz)**.** *Let $d\geq 3$ and $\mu = 1$. Let $u : I\times\mathbb{R}^d\to\mathbb{C}$ be a solution to* (1.1)*. Then*

$$\int_I\int_{\mathbb{R}^d}\frac{|u(t,x)|^{p+2}}{|x|}\,dx\,dt \lesssim \|u\|_{L_t^\infty H_x^1(I\times\mathbb{R}^d)}^2. \tag{5.9}$$

Remark 5.7. In fact, one can replace the right-hand side by

$$\||\nabla|^{1/2}u\|_{L_t^\infty L_x^2(I\times\mathbb{R}^d)}^2,$$

although we will not need this refinement.

Proof of Lemma 5.6. We will apply (5.6) with $a(x) = |x|$. In this case,

$$\nabla a = \tfrac{x}{|x|},\quad a_{jk} = \tfrac{1}{|x|}\big[\delta_{jk} - \tfrac{x_j}{|x|}\tfrac{x_k}{|x|}\big],\quad \Delta a = \tfrac{d-1}{|x|}, -\Delta\Delta a = \begin{cases} 8\pi\delta & d=3,\\ \frac{(d-1)(d-3)}{|x|^3} & d>3.\end{cases} \tag{5.10}$$

In particular,

$$\sup_{t\in I}|M_a^0(t)|\lesssim \|u\|_{L_t^\infty H_x^1(I\times\mathbb{R}^d)}^2,$$

which will give rise to the right-hand of (5.9).

To get the left-hand side of (5.9), we will establish a lower bound for $\frac{d}{dt}M_a^0$ and integrate over I. In fact, using (5.6) and (5.10)

$$\tfrac{d}{dt}M_a^0 = \int \tfrac{2\mu p(d-1)}{p+2}\tfrac{|u|^{p+2}}{|x|} + |u|^2(-\Delta\Delta a) + \tfrac{4}{|x|}|\not\nabla_0 u|^2\,dx,$$

where

$$\not\nabla_0 u = \nabla u - \nabla u\cdot\tfrac{x}{|x|}\tfrac{x}{|x|}$$

is the angular component of ∇u. In particular,

$$\int_I \tfrac{d}{dt}M_a^0\,dt \gtrsim \int_I\int_{\mathbb{R}^d}\frac{|u|^{p+2}}{|x|}\,dx\,dt,$$

giving the left-hand side of (5.9). □

The presence of the weight $|x|^{-1}$ in (5.9) means that this estimate is best-suited for preventing concentration at the origin. In particular, this result suffices to establish some global space-time bounds for radial solutions to the defocusing equation, which can only concentrate at the origin. These estimates will in turn will imply scattering in the intercritical regime (cf. Proposition 6.5).

Corollary 5.8 (Space-time bounds for radial solutions)**.** *Let $d\geq 3$, $\mu=1$, and $s_c\in(0,1)$. Let $u_0\in H^1$ be radial and let u be the corresponding global solution given by Theorem 4.2. Then*

$$\|u\|^q_{L^q_{t,x}(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|u\|^{2+\frac{2}{d-1}}_{L_t^\infty H_x^1(\mathbb{R}\times\mathbb{R}^d)} \lesssim 1, \quad \textit{where} \quad q = p+2+\tfrac{2}{d-1}.$$

Proof. First note that by uniqueness and the invariance of Δ under rotations, radial initial data lead to radial solutions; in particular, $u(t)$ is radial for each $t\in\mathbb{R}$.

The key to deducing a space-time bound from (5.9) is the following radial Sobolev embedding estimate:

$$\||x|^{\frac{d-1}{2}}f\|_{L_x^\infty(\mathbb{R}^d)} \lesssim \|f\|_{H_x^1(\mathbb{R}^d)} \quad \text{for all radial} \quad f. \tag{5.11}$$

With (5.11) in hand, we now use (5.9) to estimate

$$\begin{aligned}\iint |u|^{p+2+\frac{2}{d-1}}\,dx\,dt &\lesssim \iint \big||x|^{\frac{d-1}{2}}u\big|^{\frac{2}{d-1}}\tfrac{|u|^{p+2}}{|x|}\,dx\,dt\\ &\lesssim \||x|^{\frac{d-1}{2}}u\|^{\frac{2}{d-1}}_{L^\infty_{t,x}}\|u\|^2_{L_t^\infty H_x^1} \lesssim \|u\|^{2+\frac{2}{d-1}}_{L_t^\infty H_x^1},\end{aligned}$$

where all norms are over $\mathbb{R}\times\mathbb{R}^d$. The result now follows by recalling that the global solutions given by Theorem 4.2 are bounded in H^1. □

Exercise. Prove (5.11).

5.3. **The interaction Morawetz estimate.** As mentioned above, the Lin–Strauss Morawetz estimate is best-suited for preventing concentration at the origin. To treat non-radial solutions, it would be better to use an estimate that prevents concentration anywhere in $\mathbb{R}^d$. Such an estimate exists: it is the *interaction Morawetz inequality*, discovered originally by the 'I-team' of Colliander, Keel, Staffilani, Takaoka, and Tao.

To prove this estimate, one centers the classical Morawetz action at an arbitrary point in $\mathbb{R}^d$ and averages against the mass density. In particular, we fix $d \geq 3$ and $a(x) = |x|$. For $y \in \mathbb{R}^d$, we define

$$M_a^y(t) := 2\,\mathrm{Im} \int \bar{u}\nabla u \cdot \nabla a(x-y)\,dx,$$

and we define the interaction Morawetz action by

$$M_{int}(t) := \int M_a^y(t)|u(t,y)|^2\,dy.$$

Lemma 5.9 (Interaction Morawetz)**.** *Let $d \geq 3$, $\mu = 1$, and $a(x) = |x|$. Suppose $u : I \times \mathbb{R}^d \to \mathbb{C}$ is a solution to* (1.1)*. Then*

$$\iiint |u(t,x)|^2(-\Delta\Delta a)(x-y)|u(t,y)|^2\,dx\,dy\,dt \lesssim \|u\|^4_{L_t^\infty H_x^1(I\times\mathbb{R}^d)}. \tag{5.12}$$

Remark 5.10. In fact, one can replace the right-hand side by

$$\|u\|^2_{L_t^\infty L_x^2(I\times\mathbb{R}^d)}\||\nabla|^{1/2}u\|_{L_t^\infty L_x^2(I\times\mathbb{R}^d)},$$

but we will not need this refinement.

Proof of Lemma 5.9. We first note that

$$\sup_{t\in I}|M_{int}(t)| \lesssim \|u\|^4_{L_t^\infty H_x^1(I\times\mathbb{R}^d)},$$

which will give the right-hand side of (5.12).

As in the proof of Lemma 5.6, to get the left-hand side we will compute a lower bound for $\frac{d}{dt}M_{int}$. We compute, using (5.10), (5.2), and (5.6):

$$\begin{aligned}\tfrac{d}{dt}M_{int} &= \iint \tfrac{2\mu(d-1)p}{p+2}\tfrac{|u(y)|^2|u(x)|^2}{|x-y|} + |u(y)|^2(-\Delta\Delta a)(x-y)|u(x)|^2\,dx\,dy \\ &\quad+ \iint \tfrac{4|u(y)|^2|\not\nabla_y u(x)|^2}{|x-y|}\,dx\,dy \qquad (5.13)\\ &\quad- \iint 2\,\mathrm{Im}\,\bar{u}u_k(x)a_k(x-y)2\,\mathrm{Im}\,\partial_{y_j}(\bar{u}u_j(y))\,dx\,dy, \qquad (5.14)\end{aligned}$$

where

$$\not\nabla_y u(x) = \nabla u(x) - \nabla u(x)\cdot\tfrac{x-y}{|x-y|}\tfrac{x-y}{|x-y|}.$$

We now claim that $(5.13) + (5.14) \geq 0$. To see this, we integrate by parts, and use (5.10) and Cauchy–Schwarz to compute

$$\begin{aligned}-(5.14) &= 4\iint \mathrm{Im}(\bar{u}u_k)(x)a_{jk}(x-y)\,\mathrm{Im}(\bar{u}u_j)(y)\,dx\,dt \\ &= 4\iint \tfrac{1}{|x-y|}\big[\mathrm{Im}(\bar{u}\ \not\nabla_y u)(x)\cdot\mathrm{Im}(\bar{u}\ \not\nabla_x u)(y)\big]\,dx\,dy \\ &\leq 4\iint \tfrac{1}{2}\tfrac{|u(y)|^2|\not\nabla_y u(x)|^2}{|x-y|} + \tfrac{1}{2}\tfrac{|u(x)|^2|\not\nabla_x u(y)|^2}{|x-y|}\,dx\,dy \\ &\leq (5.13).\end{aligned}$$

Thus,

$$\tfrac{d}{dt}M_{int}(t) \geq \iint |u(y)|^2(-\Delta\Delta a)(x-y)|u(x)|^2\,dx\,dy.$$

The result follows. □

Using Lemma 5.9, we can deduce global space-time bounds for arbitrarily solutions to the defocusing equation. In particular, we have the following.

Corollary 5.11 (Space-time bounds for arbitrary solutions)**.** *Let $d \geq 3$, $\mu = 1$, and $s_c \in (0,1)$. Let $u_0 \in H^1$ and let u be the corresponding global solution to* (1.1) *given by Theorem 4.2. Then*

$$\|u\|_{L_t^{d+1} L_x^{\frac{2(d+1)}{d-1}}(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|u\|_{L_t^\infty H_x^1(\mathbb{R}\times\mathbb{R}^3)} \lesssim 1.$$

Proof. To begin, we recall that Theorem 4.2 guarantees that solutions are uniformly bounded in H^1, which is the second part of the statement.

First, using (5.10), we note that when $d = 3$ the estimate (5.12) immediately yields

$$\|u\|_{L_{t,x}^4(\mathbb{R}\times\mathbb{R}^3)} \lesssim \|u\|_{L_t^\infty H_x^1(\mathbb{R}\times\mathbb{R}^3)},$$

which is the desired result.

For $d \geq 4$, we note that

$$\mathcal{F}^{-1}(|\xi|^{-(d-3)}) = c|x|^{-3} \quad \text{for some constant} \quad c > 0. \tag{5.15}$$

Using this and Plancherel's theorem, we find

$$\begin{aligned} \text{LHS}(5.12) = \iint |u|^2 |\cdot|^{-3} * |u|^2 \,dx\,dt &\sim \iint |u|^2 |\nabla|^{-(d-3)} |u|^2 \,dx\,dt \\ &\sim \| |\nabla|^{-(\frac{d-3}{2})} |u|^2 \|_{L_{t,x}^2}^2. \end{aligned}$$

We now appeal to a technical result (Lemma A.2) and use the interaction Morawetz estimate (5.12) to get the bound

$$\| |\nabla|^{-(\frac{d-3}{4})} u\|_{L_{t,x}^4}^4 \lesssim \| |\nabla|^{-(\frac{d-3}{2})} |u|^2 \|_{L_{t,x}^2}^2 \lesssim \|u\|_{L_t^\infty H_x^1}^4.$$

Thus, in dimensions $d \geq 4$, the interaction Morawetz estimate implies control over negative order derivatives of the solution. On the other hand, in the present setting, one can also control ∇u in L_x^2. Using interpolation (see Lemma A.3), one finds

$$\|u\|_{L_t^{d+1} L_x^{\frac{2(d+1)}{d-1}}} \lesssim \| |\nabla|^{-(\frac{d-3}{4})} u\|_{L_{t,x}^4}^{\frac{4}{d+1}} \|\nabla u\|_{L_t^\infty L_x^2}^{\frac{d-3}{d+1}},$$

from which the result follows. □

Exercise. Prove (5.15).

Remark 5.12. Interaction Morawetz estimates are available in dimensions $d \in \{1, 2\}$, as well. We plan to include these estimates in a later version of the paper.

6. Intercritical NLS

In this section, we prove that scattering holds in H^1 for the defocusing intercritical NLS in dimensions $d \geq 3$; recall that intercritical refers to the restriction $0 < s_c < 1$, or equivalently $\frac{4}{d} < p < \frac{4}{d-2}$. Before stating the result, we give the precise definition of H^1 scattering.

Definition 6.1. A global solution u to (1.1) *scatters in H^1* (forward in time) if there exists a unique $u_+ \in H^1$ such that

$$\lim_{t\to\infty} \|u(t) - e^{it\Delta} u_+\|_{H_x^1(\mathbb{R}^d)} = 0.$$

Remark 6.2. Recall from Section 3 that $e^{it\Delta}u_+$ is the solution to the linear Schrödinger equation (3.1) with initial data u_+.

Remark 6.3. Of course, one can consider scattering in the backward time direction, as well, but we focus on the forward time direction for simplicity.

Theorem 6.4 (Scattering for intercritical NLS). *Let $d \geq 3$, $\mu = 1$, and $\frac{4}{d} < p < \frac{4}{d-2}$. Fix $u_0 \in H^1$ and let u be the corresponding global solution to* (1.1) *given by Theorem 4.2. Then u scatters in H^1.*

As mentioned in the introduction, the key to proving scattering is to establish decay for the solution. The following proposition will make this idea precise. We first introduce a bit of notation: for a pair of exponents (q, r), we define the scaling

$$s(q,r) := \tfrac{d}{2} - (\tfrac{2}{q} + \tfrac{d}{r}). \tag{6.1}$$

In particular, the $L_t^q L_x^r$-norm of a function scales the same way as the $L_t^\infty \dot{H}_x^{s(q,r)}$-norm under the rescaling (1.5). Note that admissible pairs (q, r) (as defined in Section 3) satisfy $s(q, r) = 0$.

Proposition 6.5 (Space-time bounds imply scattering). *Let u be as in Theorem 6.4. Suppose*

$$u \in L_t^q L_x^r(\mathbb{R} \times \mathbb{R}^d) \quad \textit{with} \quad \max\{p, 1\} \leq q < \infty \quad \textit{and} \quad s(q,r) \in (0,1). \tag{6.2}$$

Then u scatters in H^1.

Proof. First, suppose $s(q, r) = s_c$. That is, the solution obeys *critical* spacetime bounds, in the sense that the $L_t^q L_x^r$-norm is invariant under the rescaling (1.5).

We first establish that

$$\langle \nabla \rangle u \in L_t^a L_x^b(\mathbb{R} \times \mathbb{R}^d) \tag{6.3}$$

for an appropriate admissible pair (a, b). In particular, we choose a satisfying

$$\max\{\tfrac{1}{2} - \tfrac{p}{q}, 0\} \leq \tfrac{1}{a} \leq \min\{1 - \tfrac{p}{q}, \tfrac{1}{2}\}. \tag{6.4}$$

This only requires $p \leq q$ and in particular implies $2 \leq a \leq \infty$. We can therefore choose b so that (a, b) is admissible, i.e. $s(a, b) = 0$.

We next define the exponent pair (α, β) via the scaling relations

$$\tfrac{1}{\alpha} = \tfrac{p}{q} + \tfrac{1}{a}, \quad \tfrac{1}{\beta} = \tfrac{p}{r} + \tfrac{1}{b}.$$

The scaling conditions $s(q, r) = s_c$ and $s(a, b) = 0$ guarantee that $s(\alpha', \beta') = 0$. Furthermore, (6.4) guarantees that $1 \leq \alpha \leq 2$, so that (α', β') is in fact an admissible pair.

Thus, on any interval $I \subset \mathbb{R}$, we may apply Strichartz estimates (Theorem 3.1) and the chain rule to estimate

$$\begin{aligned} \|\langle \nabla \rangle u\|_{L_t^a L_x^b(I \times \mathbb{R}^d)} &\lesssim \|\langle \nabla \rangle u\|_{L_t^\infty L_x^2(I \times \mathbb{R}^d)} + \|\langle \nabla \rangle (|u|^p u)\|_{L_t^\alpha L_x^\beta(I \times \mathbb{R}^d)} \\ &\lesssim \|\langle \nabla \rangle u\|_{L_t^\infty L_x^2(I \times \mathbb{R}^d)} + \|u\|_{L_t^q L_x^r(I \times \mathbb{R}^d)}^p \|\langle \nabla \rangle u\|_{L_t^a L_x^b(I \times \mathbb{R}^d)}. \end{aligned}$$

Now fix $\varepsilon > 0$. Recalling (6.2) (and in particular that $q < \infty$), we can break $\mathbb{R}$ into finitely many intervals I such that

$$\|u\|_{L_t^q L_x^r(I \times \mathbb{R}^d)}^p < \varepsilon$$

on each interval. Choosing ε sufficiently small, the above estimate implies

$$\|\langle \nabla \rangle u\|_{L_t^a L_x^b(I \times \mathbb{R}^d)} \lesssim \|\langle \nabla \rangle u\|_{L_t^\infty L_x^2(I \times \mathbb{R}^d)} \lesssim 1$$

on each interval, which in turn implies (6.3).

We can now establish scattering by showing that $\{e^{-it\Delta}u(t) : t \in \mathbb{R}\}$ is Cauchy in H^1 as $t \to \infty$. Indeed, estimating via Strichartz as above,

$$\|e^{-it\Delta}u(t) - e^{-is\Delta}u(s)\|_{H^1_x(\mathbb{R}^d)} \lesssim \|u\|^p_{L^q_t L^r_x((s,t)\times\mathbb{R}^d)} \|\langle\nabla\rangle u\|_{L^a_t L^b_x((s,t)\times\mathbb{R}^d)} \to 0$$

as $s, t \to \infty$. Again, we rely on the fact that $q < \infty$.

We now turn to the general case $s(q, r) \in (0, 1)$. We recall that by uniform H^1-boundedness, we have

$$u \in L^\infty_t L^2_x \quad \text{and} \quad u \in L^\infty_t L_x^{\frac{2d}{d-2}};$$

indeed, the second bound follows from Sobolev embedding. Noting that $s(\infty, 2) = 0$ and $s(\infty, \frac{2d}{d-2}) = 1$, we deduce by interpolation with $u \in L^q_t L^r_x$ that $u \in L^{q_c}_t L^{r_c}_x(\mathbb{R} \times \mathbb{R}^d)$ for some (q_c, r_c) satisfying $s(q_c, r_c) = s_c$ (see the exercise below). Moreover, the interpolation guarantees that $p \leq q < q_c < \infty$. Thus we have reduced to the first case described above, so that scattering follows. □

Exercise. Suppose that $u \in L^{q_j}_t L^{r_j}_x$, with $1 \leq q_j, r_j \leq \infty$, for $j \in \{1, 2\}$. Denote $s_j = s(q_j, r_j)$ and suppose without loss of generality that $s_1 < s_2$. Prove that for any $s \in (s_1, s_2)$, there exists $1 \leq q, r \leq \infty$ with $s(q, r) = s$ such that $u \in L^q_t L^r_x$.

We can now quickly dispense with the proof of Theorem 6.4.

Proof of Theorem 6.4. By the interaction Morawetz inequality, specifically Corollary 5.11, we have $u \in L^{d+1}_t L_x^{\frac{2(d+1)}{d-1}}(\mathbb{R} \times \mathbb{R}^d)$. Noting that

$$s(d+1, \tfrac{2(d+1)}{d-1}) = \tfrac{d-2}{d+1} \in (0, 1)$$

and $p < d+1$ for all p under consideration, Proposition 6.5 implies that the solution scatters in H^1. □

Remark 6.6. Note that the Lin–Strauss Morawetz estimate (specifically, Corollary 5.8) implies scattering in the radial case. Indeed, one gets $u \in L^q_{t,x}(\mathbb{R} \times \mathbb{R}^d)$ with $q = p + 2 + \frac{2}{d-1}$. Clearly $p < q$, while a bit of computation shows that

$$s(q, q) \in (0, 1) \iff \tfrac{4}{d} - \tfrac{2}{d-1} < p < \tfrac{4}{d-2}\big(1 + \tfrac{d}{2d-2}\big).$$

In our presentation of the proof of scattering, it is not clear that the radial case is any 'easier'. To see where the radial assumption actually simplifies matters, compare the proofs of Corollary 5.8 and Corollary 5.11.

7. Mass-subcritical NLS

In this section we discuss the mass-subcritical case, which refers to $s_c < 0$ or equivalently $p < \frac{4}{d}$. As discussed in Section 4, solutions to (1.1) are global (even in the focusing case) for any initial data in L^2_x. However, we will consider only the defocusing problem and data in the weighted space Σ defined in (1.9), as this assumption gives us access to the pseudoconformal energy estimate (Lemma 5.4).

We will consider scattering in two different topologies.

Definition 7.1. A global solution u to (1.1) *scatters in* L^2 (forward in time) if there exists a unique $u_+ \in L^2$ such that

$$\lim_{t\to\infty} \|u(t) - e^{it\Delta}u_+\|_{L^2_x} = 0.$$

The solution *scatters in* Σ (forward in time) if there exists unique $u_+ \in \Sigma$ such that

$$\lim_{t\to\infty} \|e^{-it\Delta}u(t) - u_+\|_\Sigma = 0.$$

Remark 7.2. It is not clear whether scattering in Σ is equivalent to the statement that

$$\lim_{t\to\infty} \|u(t) - e^{it\Delta}u_+\|_\Sigma = 0.$$

For this question, we refer the reader to papers of Bégout, who has some positive results under restrictions on d and p.

The main nonlinear tool we have for studying the scattering theory in weighted spaces is the pseudoconformal energy estimate, Lemma 5.4. In particular, this estimate gives a decay rate of $t^{-\frac{dp}{2}}$ for the potential energy. This power of t is integrable as $t \to \infty$ precisely when $p > \frac{2}{d}$. Thus, one is led to suspect that if $p > \frac{2}{d}$, the nonlinear effects become negligible as $t \to \infty$, while if $0 < p \le \frac{2}{d}$ the nonlinearity has some net effect as $t \to \infty$. In particular, $p = \frac{2}{d}$ becomes a natural candidate for the threshold for whether or not scattering occurs.

In fact, we will prove positive scattering results in the range $\frac{2}{d} < p < \frac{4}{d}$ (known as the *short range* case); we will prove that no (linear) scattering is possible when $0 < p < \frac{2}{d}$ (known as the *long range* case).

7.1. **The short-range case.** We first present a result of Tsutsumi and Yajima, which establishes scattering in L^2_x in the short range case.

Theorem 7.3 (L^2-scattering)**.** *Let* $\frac{2}{d} < p < \frac{4}{d}$ *and* $\mu = 1$. *Let* $u_0 \in \Sigma$ *and take* u *to be the corresponding global solution to* (1.1) *given by Theorem 4.2. Then* u *scatters in* L^2.

The drawback of this result is that while initial data is taken from $u_0 \in \Sigma$, scattering is only shown to hold in L^2. The strength of this result is that it treats all possible p for which linear scattering is possible (cf. Theorem 7.7 below).

Proof of Theorem 7.3. We take advantage of the fact that the asymptotics of the linear Schrödinger equation in the L^2-topology are slightly simplified. In particular, using Lemma 3.2, we see that to prove scattering in L^2, it suffices to show that there exists unique $W \in L^2$ such that

$$\lim_{t\to\infty} \|u(t) - M(t)D(t)W\|_{L^2_x} = 0, \tag{7.1}$$

where we recall the notation from (3.7). Indeed, then scattering in L^2 holds with $u_+ = \mathcal{F}^{-1}W$.

To this end, we define the function $w(t)$ via

$$u(t) := M(t)D(t)w(t) \tag{7.2}$$

and endeavor to show that w has a limit in L^2 as $t \to \infty$.

For this, we first note that as u solves (1.1), the function w solves the equation

$$(i\partial_t + \tfrac{1}{2t^2}\Delta)w = (2t)^{-\frac{dp}{2}}|w|^p w. \tag{7.3}$$

We can also translate the result of the pseudoconformal energy estimate (Lemma 5.4) into information about w, namely:

$$\|\nabla w(t)\|_{L^2_x} \lesssim t^{1-\frac{dp}{4}} \quad \text{and} \quad \|w(t)\|_{L^{p+2}_x} \lesssim 1 \quad \text{for} \quad t \ge 1. \tag{7.4}$$

Note also that by conservation of mass, $w(t)$ is uniformly bounded in L^2.

We first show that $w(t)$ converges weakly in L^2 as $t \to \infty$. By L^2-boundedness, it suffices to test against a Schwartz function φ (say). Using Hölder's inequality, integration by parts, (7.3), (7.4), and the condition $p > \frac{2}{d}$, we estimate

$$\begin{aligned}
|\langle w(t) - w(s), \varphi\rangle| &\lesssim \int_s^t \tau^{-2}|\langle \Delta w(\tau), \varphi\rangle| + \tau^{-\frac{dp}{2}}|\langle (|w|^p w)(\tau), \varphi\rangle|\, d\tau \\
&\lesssim \int_s^t \tau^{-2}\|\nabla w(\tau)\|_{L_x^2}\|\nabla\varphi\|_{L_x^2} + \tau^{-\frac{dp}{2}}\|w(\tau)\|_{L_x^{p+2}}^{p+1}\|\varphi\|_{L_x^{p+2}}\, d\tau \\
&\lesssim \int_s^t \tau^{-1-\frac{dp}{4}} + \tau^{-\frac{dp}{2}}\, d\tau \to 0 \quad \text{as} \quad s, t \to \infty.
\end{aligned}$$

In particular, there exists a unique W in L^2 so that $w(t)$ converges to W weakly in L^2 as $t \to \infty$.

We now upgrade to strong convergence. To begin, we use weak convergence to note

$$\lim_{t\to\infty} \langle w(t) - W, w(t) - W\rangle = \lim_{t\to\infty}\lim_{s\to\infty} \langle w(t) - w(s), w(t)\rangle.$$

We now estimate similarly to the above:

$$\begin{aligned}
|\langle w(t) - w(s), w(t)\rangle| &\lesssim \int_t^s \tau^{-2}\|\nabla w(\tau)\|_{L_x^2}\|\nabla w(t)\|_{L_x^2} \\
&\qquad + \tau^{-\frac{dp}{2}}\|w(\tau)\|_{L_x^{p+2}}^{p+1}\|w(t)\|_{L_x^{p+2}}\, d\tau \\
&\lesssim \int_s^t \tau^{-1-\frac{dp}{4}} t^{1-\frac{dp}{4}} + \tau^{-\frac{dp}{2}}\, d\tau \to 0 \quad \text{as} \quad s, t \to \infty.
\end{aligned}$$

In particular $w(t) \to W$ strongly in L^2, which completes the proof. □

Remark 7.4. The original proof uses the pseudoconformal transformation, which is not discussed in this note but is closely related to the transformation $u \mapsto w$ in (7.2). The author has never been able to find a proof of Theorem 7.3 that proceeds directly by showing $\{e^{-it\Delta}u(t)\}$ has a strong limit in L_x^2 as $t \to \infty$.

One curious feature of the proof of Theorem 7.3 is that one does not rely on critically-scaling global space-time bounds for the solutions. This is in contrast to the 'modern' approach to scattering; see, for example, the proofs in Section 6.

In fact, it is a bit remarkable that one can address the full range $p > \frac{2}{d}$, which corresponds to $s_c > -\frac{d}{2}$, in light of the fact that the lowest 'regularity' associated with the space Σ is $s_c = -1$. Of course, scattering is only shown to hold in the L^2-topology.

If one did have critical space-time bounds, one could expect scattering to hold in the Σ topology (at least for $s_c > -1$, i.e. $p > \frac{4}{d+2}$). In fact, the pseudoconformal energy estimate (Lemma 5.4) does imply some space-time bounds for the solution:

$$\|u(t)\|_{L_x^{p+2}} \lesssim t^{-\frac{dp}{2(p+2)}} \implies u \in L_t^q L_x^{p+2} \quad \text{for all} \quad \tfrac{2(p+2)}{dp} < q \leq \infty.$$

(Strictly speaking, we use H^1 bounds and the Gagliardo–Nirenberg inequality to control the L^{p+2}-norm near $t = 0$.) Note that the scaling associated to these spaces is given by

$$s(q, p+2) \in \left(\tfrac{-dp}{2(p+2)}, \tfrac{dp}{2(p+2)}\right) \quad \text{for} \quad \tfrac{2(p+2)}{dp} < q \leq \infty,$$

where we recall the notation from (6.1). Thus, the pseudoconformal energy estimate implies critical space-time bounds for the solution whenever

$$-\tfrac{dp}{2(p+2)} < s_c = -(\tfrac{2}{p} - \tfrac{d}{2}), \quad \text{i.e.} \quad p > \alpha(d) := \tfrac{2-d+\sqrt{d^2+12d+4}}{2d}.$$

The exponent $\alpha(d)$ is known as the *Strauss exponent*. As the preceding discussion suggests, scattering in Σ holds above the exponent. In particular, we have the following result due to Cazenave and Weissler.

Theorem 7.5 (Scattering in weighted spaces)**.** *Let $d \geq 3$, $\alpha(d) < p < \frac{4}{d}$, and $\mu = 1$. Let $u_0 \in \Sigma$ and let u be the corresponding solution to* (1.1) *given by Theorem 4.2. Then u scatters in Σ.*

Proof. We argue as in Proposition 6.2. By the discussion preceding the statement of Theorem 7.5, the assumption $p > \alpha(d)$ and pseudoconformal energy estimate (Lemma 5.4) imply the following critical space-time bounds for the solution u:

$$u \in L_t^q L_x^{p+2}(\mathbb{R} \times \mathbb{R}^d), \quad q := \tfrac{2p(p+2)}{4-p(d-2)}.$$

Arguing as in Proposition 6.2, we may find an admissible pair (a, b) and a dual admissible pair (α, β) satisfying the scaling relations

$$\tfrac{1}{\alpha} = \tfrac{p}{q} + \tfrac{1}{a}, \quad \tfrac{1}{\beta} = \tfrac{p}{p+2} + \tfrac{1}{b}. \tag{7.5}$$

This only requires $p \leq q$, which is equivalent to $dp > 0$. We now claim that

$$Ju,\ \langle\nabla\rangle u \in L_t^a L_x^b(\mathbb{R} \times \mathbb{R}^d),$$

where we recall J is as in (3.8). Indeed, the proof that $\langle\nabla\rangle u \in L_t^a L_x^b$ is exactly like the proof of (6.3). The proof that $Ju \in L_t^a L_x^b$ will be similar, once we observe that J essentially obeys a chain rule and that we can establish uniform bounds for Ju in L^2. We turn to the details.

First, to shorten formulas we introduce the notation

$$\|u\|_{X(I)} = \|u\|_{L_t^\infty L_x^2(I\times\mathbb{R}^d)} + \|u\|_{L_t^a L_x^b(I\times\mathbb{R}^d)}.$$

Now fix $\varepsilon > 0$ and (noting $q < \infty$) divide $\mathbb{R}$ into finitely many intervals $I = [t_0, t_1]$ such that

$$\|u\|_{L_t^q L_x^{p+2}(I\times\mathbb{R}^d)}^p < \varepsilon.$$

Using (3.9) and Strichartz estimates (Theorem 3.1), we first note

$$\|J(t)e^{i(t-t_0)\Delta}u(t_0)\|_{X(I)} \lesssim \|J(t_0)u(t_0)\|_{L_x^2}. \tag{7.6}$$

For the nonlinearity, we use (3.9), the gauge-invariance of $|u|^p u$, and (7.5) to estimate

$$\|J(|u|^p u)\|_{L_x^\beta} = \|2it\nabla\big(|M(-t)u|^p M(-t)u\big)\|_{L_x^\beta} \lesssim \|u\|_{L_x^{p+2}}^p \|Ju\|_{L_x^b}. \tag{7.7}$$

Thus, using $J(t)e^{i(t-s)\Delta} = e^{i(t-s)\Delta}J(s)$, (7.6), Strichartz estimates (Theorem 3.1), and (7.5), we can estimate

$$\begin{aligned}\|Ju\|_{X(I)} &\lesssim \|J(t_0)u(t_0)\|_{L_x^2} + \|J(|u|^p u)\|_{L_t^\alpha L_x^\beta(I\times\mathbb{R}^d)} \\ &\lesssim \|J(t_0)u(t_0)\|_{L_x^2} + \|u\|_{L_t^q L_x^{p+2}(I\times\mathbb{R}^d)}^p \|Ju\|_{L_t^a L_x^b(I\times\mathbb{R}^d)} \\ &\lesssim \|J(t_0)u(t_0)\|_{L_x^2} + \varepsilon\|Ju\|_{X(I)}.\end{aligned}$$

Thus

$$\|Ju\|_{L_t^\infty L_x^2([t_0,t_1]\times\mathbb{R}^d)} + \|Ju\|_{L_t^a L_x^b([t_0,t_1]\times\mathbb{R}^d)} \lesssim \|J(t_0)u(t_0)\|_{L_x^2}.$$

Starting at $t_0 = 0$, say, this allows us to deduce that

$$\|Ju\|_{L_t^a L_x^b(I\times\mathbb{R}^d)} \lesssim 1$$

uniformly on each interval. This implies $Ju \in L_t^a L_x^b(\mathbb{R}\times\mathbb{R}^d)$.

We can now deduce scattering in Σ. Indeed, scattering in H^1 follows as in the proof of Proposition 6.2. For the weighted term, we estimate as above to find

$$\begin{aligned}\|xe^{-it\Delta}u(t) - xe^{-is\Delta}u(s)\|_{L_x^2} &\lesssim \|J(|u|^p u)\|_{L_t^\alpha L_x^\beta((s,t)\times\mathbb{R}^d)} \\ &\lesssim \|u\|_{L_t^q L_x^{p+2}((s,t)\times\mathbb{R}^d)} \|Ju\|_{L_t^a L_x^b((s,t)\times\mathbb{R}^d)} \to 0\end{aligned}$$

as $s,t\to\infty$. □

Remark 7.6. It is possible to prove scattering in Σ down to $p > \frac{4}{d+2}$ for sufficiently small initial data. In this case, one can use Strichartz estimates and a bootstrap argument as in the preceding argument to prove control over a certain critically-scaling Lorentz-space modified space-time norm, which in turn implies scattering. We plan to include this result in a later version of the paper. The question of scattering in Σ below the Strauss exponent for arbitrary data is still open.

7.2. **The long-range case.** We next turn to the case $0 < p \leq \frac{2}{d}$ and prove that scattering in L^2 only occurs for the trivial zero solution. As in the proof of Theorem 7.3, key ingredients include the estimates for nonlinear solutions given by the pseudoconformal energy estimate (Lemma 5.4) and the Fraunhofer formula describing the asymptotics of the linear Schrödinger equation (Lemma 3.2).

The following result is due to Strauss and Barab.

Theorem 7.7 (No L^2-scattering in the long-range case)**.** *Let $0 < p \leq \frac{2}{d}$ and $\mu = 1$. Let $u_0 \in \Sigma$ and take u to be the corresponding global solution to* (1.1) *given by Theorem 4.2. If u scatters in L^2, then $u \equiv 0$.*

Proof. Suppose u scatters to some function u_+ in L^2, i.e.

$$\lim_{t\to\infty} \|u(t) - e^{it\Delta}u_+\|_{L_x^2} = 0. \tag{7.8}$$

Suppose towards a contradiction that $u_0 \neq 0$. Note that

$$\|u_0\|_{L_x^2} = \|u_+\|_{L_x^2} > 0.$$

Fix $\varepsilon > 0$. By density, we can find a Schwartz function ϕ such that

$$\|u_+ - \phi\|_{L_x^2} < \varepsilon.$$

In particular, choosing ε sufficiently small, we can guarantee that $\phi \neq 0$.

Using (7.8), we find that for $T = T(\varepsilon)$ sufficiently large, we have

$$\|u(t) - e^{it\Delta}\phi\|_{L_x^2} \leq 2\varepsilon \quad \text{for all} \quad t \geq T. \tag{7.9}$$

Set $v(t) = e^{it\Delta}\phi$; in particular v solves $(i\partial_t + \Delta)v = 0$. Note that

$$\sup_{t\geq T} |\langle u(t), v(t)\rangle| \leq 2\|u_0\|_{L_x^2}^2, \tag{7.10}$$

where $\langle\cdot,\cdot\rangle$ denotes the L^2 inner product.

Using (1.1), we can compute

$$i\partial_t\langle u, v\rangle = \int |u|^p u\bar{v}\,dx = \int |v|^{p+2}\,dx + \int \big(|u|^p u - |v|^p v\big)\bar{v}\,dx.$$

for each $t \geq T$. We will establish a lower bound for the first integral and an upper bound for the second integral that will lead to a contradiction to (7.10).

To begin, we fix $k > 0$ and use Hölder's inequality to deduce the lower bound

$$\left(\int |v(t)|^{p+2} \, dx \right)^{\frac{1}{p+2}} \gtrsim \left(\int_{|x| \leq kt} |v(t)|^2 \, dx \right)^{\frac{1}{2}} t^{-\frac{dp}{2(p+2)}}.$$

We now note that as $\phi \neq 0$, for $k = k(\phi)$ sufficiently large, we can guarantee that

$$\int_{|\xi| \leq k} |\widehat{\phi}(\xi)|^2 \, d\xi \gtrsim 1,$$

so that by the Fraunhofer formula (Lemma 3.2) and a change of variables, we can choose T possibly even larger to guarantee that

$$\int |v(t)|^{p+2} \, dx \gtrsim t^{-\frac{dp}{2}} \quad \text{for all} \quad t \geq T.$$

On the other hand, using Hölder's inequality, the dispersive estimate (3.6), the pseudoconformal energy estimate (Lemma 5.4), and (7.9), we have

$$\begin{aligned} \left| \int (|u|^p u - |v|^p v) \bar{v} \, dx \right| &\lesssim \|(u^p + v^p)(u - v) v\|_{L_x^1} \\ &\lesssim \|u - v\|_{L_x^2} (\|u\|_{L_x^{p+2}}^p + \|v\|_{L_x^{p+2}}^p) \|v\|_{L_x^{\frac{2(p+2)}{2-p}}} \\ &\lesssim \varepsilon t^{-\frac{dp}{2}} \end{aligned}$$

for all $t \geq T$.

In particular, choosing ε sufficiently small, we can combine the estimates above to deduce

$$|\langle u(t), v(t) \rangle - \langle u(T), v(T) \rangle| \gtrsim \int_T^t \tau^{-\frac{dp}{2}} \, d\tau$$

for all $t \geq T$. However, as $p \leq \frac{2}{d}$, the integral on the right-hand diverges as $t \to \infty$; this contradicts (7.10) and completes the proof. □

8. Modified scattering

It is an interesting question to describe the long-time asymptotics of solutions to (1.1) in the long range case $0 < p \leq \frac{2}{d}$. So far, the only results available are in the borderline case $p = \frac{2}{d}$ in dimensions $d \in \{1, 2, 3\}$. In the special case $p = 2$ in $d = 1$, the equation is completely integrable and can be treated via inverse scattering techniques, even for large data. Otherwise, results are restricted to the small data regime. The restriction on dimension is easy to explain: the critical regularity associated to $p = \frac{2}{d}$ is $s_c = -\frac{d}{2}$, and the analysis generally requires one to take $|s_c|$ derivatives of the nonlinearity $|u|^{\frac{2}{d}} u$. Thus, the analysis breaks down if $\frac{d}{2} > 1 + \frac{2}{d}$, i.e. $d > 3$.

We focus on the particular case $d = 1$ and $p = 2$ for the sake of simplicity, but the arguments generalize easily to $d \in \{2, 3\}$. We consider initial data in Σ; in general, the sharpest results available consider initial data in the space Σ^γ defined via the norm

$$\|f\|_{\Sigma^\gamma}^2 = \|\langle \nabla \rangle^\gamma f\|_{L_x^2}^2 + \||x|^\gamma f\|_{L_x^2} \quad \text{for} \quad \gamma > \tfrac{d}{2}.$$

The proof is based on studying the dynamics of $\mathcal{F} e^{-it\Delta} u(t)$, which approximately solves an ODE. One can show that solutions to the ODE remain bounded by using an appropriate integrating factor; it is the presence of this integrating factor that leads to the modification to linear asymptotics. To prove that the ODE accurately models the PDE requires good bounds on the solution, including L_x^∞-bounds decaying like $t^{-\frac{d}{2}}$. Note that this rate matches that of solutions to the linear equation; cf. (3.5). One also proves control over Ju in L_x^2, showing in particular that this quantity grows like a very small power of t. In the general case, one controls a power of J, namely J^γ for $\gamma > \frac{d}{2}$; note that the identities (3.9) suggest how to define powers of J. In practice, the requisite bounds are proven via a bootstrap argument, where a small data assumption allows for the estimates to close.

We turn to the details. The following theorem is due to Hayashi and Naumkin, but alternate proofs (ultimately of a similar spirit) have been given by Kato and Pusateri, and by Ifrim and Tataru. Our presentation is closest to that of Hayashi and Naumkin.

Theorem 8.1 (Modified scattering)**.** *Let $d = 1$, $p = 2$ and $\mu \in \{\pm 1\}$. Let $u_0 \in \Sigma$, and let u be the corresponding global solution to* (1.1). *If $\|u_0\|_\Sigma$ is sufficiently small, then there exists $W \in L^\infty$ so that*

$$\lim_{t\to\infty} \|u(t) - M(t)D(t)e^{-i\frac{\mu}{2}|W|^2 \log t} W\|_{L_x^\infty} = 0,$$

where $M(t)$ and $D(t)$ are as in (3.7).

Remark 8.2. Comparing with Lemma 3.2, we see that there is only a modification in the phase compared to linear scattering. We also note that the convergence also holds in L_x^2, which we leave as an exercise to the motivated reader.

Proof. Denote $\varepsilon = \|u_0\|_\Sigma$ and let $\delta > 0$ to be determined below.

The proof is based off of a bootstrap argument controlling two norms of the solution, namely,

$$\|u(t)\|_X := \sup_{s\in[0,t]} \big\{\|u(s)\|_{L_x^2} + \langle s\rangle^{-\delta}\|J(s)u(s)\|_{L_x^2}\big\}, \quad \|u(t)\|_S := \sup_{s\in[0,t]} \langle s\rangle^{\frac{1}{2}}\|u(s)\|_{L_x^\infty},$$

where J is as in (3.8). Note that by conservation of mass, the L^2-component of the X-norm is clearly under control. Furthermore, by local theory and the Sobolev embedding $L_x^\infty \subset H_x^1$, one has

$$\sup_{t\in[0,1]} \big\{\|u(t)\|_X + \|u(t)\|_S\big\} \lesssim \varepsilon. \tag{8.1}$$

Applying the 'chain rule' for J as in (7.7), we can estimate

$$\|Ju(t)\|_{L_x^2} \lesssim \varepsilon + \int_1^t s\|u(s)\|_{L_x^\infty}^2 \|Ju(s)\|_{L_x^2}\,\frac{ds}{s},$$

and hence by Gronwall's inequality, we can deduce

$$\|u(t)\|_X \lesssim \varepsilon\langle t\rangle^{C\|u(t)\|_S^2} \tag{8.2}$$

for some absolute constant $C > 0$ and for all $t \geq 1$. This shows that control over the S-norm gives control over the X-norm. To close a bootstrap argument, we will prove a converse to this.

In particular, we now aim to control the solution in L_x^∞. To this end, we fix $t \geq 1$ and use (3.7) to write

$$u(t) = M(t)D(t)\mathcal{F}e^{-it\Delta}u(t) + M(t)D(t)\mathcal{F}[M(t) - 1]e^{-it\Delta}u(t). \tag{8.3}$$

In the second term, we can exhibit some additional decay from the factor of $M(t)-1$; indeed,

$$|M(t)-1| \leq |t|^{-\frac{1}{5}}|x|^{\frac{2}{5}}. \tag{8.4}$$

Thus, using the Hausdorff–Young and Cauchy–Schwarz inequalities,

$$\begin{aligned}\|M(t)D(t)\mathcal{F}[M(t)-1]e^{-it\Delta}u(t)\|_{L_x^\infty} &\lesssim t^{-\frac{1}{2}}t^{-\frac{1}{5}}\||x|^{\frac{2}{5}}e^{-it\Delta}u(t)\|_{L_x^1} \\ &\lesssim t^{-\frac{1}{2}}t^{-\frac{1}{5}}\|\langle x\rangle^{-1+\frac{2}{5}}\|_{L_x^2}\|\langle x\rangle e^{-it\Delta}u(t)\|_{L_x^2} \\ &\lesssim t^{-\frac{1}{2}}t^{-\frac{1}{5}}\langle t\rangle^{\delta}\|u(t)\|_X.\end{aligned}$$

We now turn to the first term in (8.3). We introduce the notation

$$f(t)=\mathcal{F}e^{-it\Delta}u(t) \quad \text{and} \quad \tilde{f}(t)=\mathcal{F}M(t)e^{-it\Delta}u(t).$$

Using (1.1), we compute

$$i\partial_t f = \tfrac{\mu}{2t}|f|^2 f + \tfrac{\mu}{2t}\big\{\mathcal{F}[M(-t)-1]\mathcal{F}^{-1}|\tilde{f}|^2\tilde{f} + |\tilde{f}|^2\tilde{f} - |f|^2 f\big\}.$$

We wish to remove the first term; thus, we introduce the integrating factor

$$B(t)=\exp\bigg\{i\mu\int_1^t |f(s)|^2\tfrac{ds}{2s}\bigg\} \quad \text{and define} \quad g(t)=B(t)f(t).$$

We then have

$$i\partial_t g = \tfrac{\mu}{2t}B(t)\big\{\mathcal{F}[M(-t)-1]\mathcal{F}^{-1}|\tilde{f}|^2\tilde{f} + |\tilde{f}|^2\tilde{f} - |f|^2 f\big\}. \tag{8.5}$$

We now estimate as above to get

$$\begin{aligned}\|\mathcal{F}[M(-t)-1]\mathcal{F}^{-1}|\tilde{f}|^2\tilde{f}\|_{L_x^\infty} &\lesssim t^{-\frac{1}{5}}\||x|^{\frac{2}{5}}\mathcal{F}^{-1}|\tilde{f}|^2\tilde{f}\|_{L_x^1} \\ &\lesssim t^{-\frac{1}{5}}\|\langle\nabla\rangle(|\tilde{f}|^2\tilde{f})\|_{L_x^2} \\ &\lesssim t^{-\frac{1}{5}}\|\tilde{f}\|_{L_x^\infty}^2\|\langle\nabla\rangle\tilde{f}\|_{L_x^2}.\end{aligned}$$

Noting that by Cauchy–Schwarz,

$$\|\tilde{f}(t)\|_{L_x^\infty} \lesssim \|e^{-it\Delta}u(t)\|_{L_x^1} \lesssim \|\langle x\rangle e^{-it\Delta}u(t)\|_{L_x^2} \lesssim \langle t\rangle^{\delta}\|u(t)\|_X, \tag{8.6}$$

and that by Plancherel

$$\|\langle\nabla\rangle\tilde{f}\|_{L_x^2} \lesssim \|\langle x\rangle e^{-it\Delta}u(t)\|_{L_x^2} \lesssim \langle t\rangle^{\delta}\|u(t)\|_X,$$

we have

$$\|\mathcal{F}[M(-t)-1\mathcal{F}^{-1}|\tilde{f}|^2\tilde{f}\|_{L_x^\infty} \lesssim t^{-\frac{1}{5}}\langle t\rangle^{3\delta}\|u(t)\|_X^3.$$

Next, we note that

$$\tilde{f}(t)-f(t)=\mathcal{F}[M(t)-1]e^{-it\Delta}u(t).$$

Thus, using (8.6) (and noting that the same bound holds for f), we can estimate as above to find

$$\begin{aligned}\|\,|\tilde{f}|^2\tilde{f}-|f|^2 f\|_{L_x^\infty} &\lesssim \langle t\rangle^{2\delta}\|u(t)\|_X^2\|\mathcal{F}[M(t)-1]e^{-it\Delta}u\|_{L_x^\infty} \\ &\lesssim t^{-\frac{1}{5}}\langle t\rangle^{3\delta}\|u(t)\|_X^3.\end{aligned}$$

Continuing from (8.5), we find

$$\|\partial_t g(t)\|_{L_x^\infty} \lesssim t^{-1}t^{-\frac{1}{5}}\langle t\rangle^{3\delta}\|u(t)\|_X^3. \tag{8.7}$$

In particular, noting that $|g(t)|=|f(t)|$ and using (8.1), we deduce

$$\|f(t)\|_{L_x^\infty} \lesssim \varepsilon + \int_1^t s^{-1}s^{-\frac{1}{5}}\langle s\rangle^{3\delta}\|u(s)\|_X^3\,ds.$$

Continuing from (8.3) we get the estimate

$$\|u(t)\|_S \lesssim \varepsilon + t^{-\frac{1}{5}}\langle t\rangle^{\delta}\|u(t)\|_X + \int_1^t s^{-1}s^{-\frac{1}{5}}\langle s\rangle^{3\delta}\|u(s)\|_X^3\,ds \tag{8.8}$$

for $t \geq 1$.

Choosing δ sufficiently small (say $\delta = \frac{1}{30}$) and ε sufficiently small, a continuity argument using (8.2) and (8.8) now implies

$$\sup_{t\in[0,\infty)} \big\{\|u(t)\|_X + \|u(t)\|_S\big\} \lesssim \varepsilon. \tag{8.9}$$

With (8.9) in hand, we can now establish the asymptotics. First, using (8.9) and (8.7) (recalling that we set $\delta = \frac{1}{30}$), we note that

$$\|\partial_t g(t)\|_{L_x^\infty} \lesssim t^{-1}t^{-\frac{1}{10}} \quad \text{and} \quad \|g\|_{L_{t,x}^\infty} \lesssim 1. \tag{8.10}$$

This implies that $\{g(t) : t \geq 0\}$ is Cauchy in L_x^∞ as $t \to \infty$, which implies that there exists $W_0 \in L^\infty$ so that

$$\lim_{t\to\infty} \|g(t) - W_0\|_{L_x^\infty} = \lim_{t\to\infty} \|\mathcal{F}e^{-it\Delta}u(t) - B(t)^{-1}W_0\|_{L_x^\infty} = 0.$$

Using (8.9) in conjunction with the estimates given above, we can show that

$$\lim_{t\to\infty} \|\mathcal{F}e^{-it\Delta}u(t) - (M(t)D(t))^{-1}u(t)\|_{L_x^\infty} = 0.$$

Thus, we can deduce that

$$\lim_{t\to\infty} \|u(t) - M(t)D(t)B(t)^{-1}W_0\|_{L_x^\infty} = 0. \tag{8.11}$$

We now take a closer look at $B(t)^{-1}$. We first recall $|f| = |g|$, and we define $\Psi(t)$ via

$$\int_1^t |g(s)|^2 \tfrac{ds}{2s} =: \tfrac{1}{2}|g(t)|^2 \log t + \Psi(t). \tag{8.12}$$

We claim that $\{\Psi(t) : t \geq 0\}$ is Cauchy in L_x^∞ as $t \to \infty$. In fact, a bit of rearranging shows

$$\Psi(t) - \Psi(s) = \int_s^t \big(|g(\tau)|^2 - |g(t)|^2\big)\tfrac{d\tau}{2\tau} + \tfrac{1}{2}\big(|g(s)|^2 - |g(t)|^2\big)\log s,$$

and as (8.10) implies

$$\big||g(t_2)|^2 - |g(t_1)|^2\big| \lesssim t_1^{-\frac{1}{10}} \quad \text{for} \quad t_2 > t_1 > 1,$$

the claim follows. In particular, there exists $\Phi \in L^\infty$ so that

$$\lim_{t\to\infty} \Psi(t) = \Phi.$$

Continuing from (8.11), we deduce that

$$\lim_{t\to\infty} \|u(t) - M(t)D(t)e^{-i\frac{\mu}{2}|W_0|^2\log t - i\frac{\mu}{2}\Phi}W_0\|_{L_x^\infty} = 0.$$

The result now follows with $W = e^{-i\frac{\mu}{2}\Phi}W_0$. □

Exercise. Show that the convergence in Theorem 8.1 holds in L_x^2, as well.

Appendix A. A few technical results

We record here a few technical harmonic analysis results. We will need the standard Littlewood–Paley projections P_N, where $N \in 2^{\mathbb{Z}}$. These are defined as Fourier multiplier operators $P_N = \mathcal{F}^{-1}\varphi(\frac{\xi}{N})\mathcal{F}$, where φ is a smooth cutoff to an annulus $|\xi| \sim 1$.

The primary technical tool we will use in this section is the Littlewood–Paley square function estimate.

Lemma A.1 (Square function estimate)**.** *For any* $1 < r < \infty$,

$$\|f\|_{L_x^r(\mathbb{R}^d)} \sim \|Sf\|_{L_x^r(\mathbb{R}^d)}, \quad Sf(x) := \Big(\sum_{N\in 2^{\mathbb{Z}}} |P_N f(x)|^2\Big)^{\frac{1}{2}}.$$

More generally, for $1 < r < \infty$ *and* $s \in \mathbb{R}$,

$$\||\nabla|^s f\|_{L_x^r(\mathbb{R}^d)} \sim \Big\|\Big(\sum_N N^{2s}|P_N f(x)|^2\Big)^{\frac{1}{2}}\Big\|_{L_x^r(\mathbb{R}^d)}.$$

The following technical lemma helps deduce a suitable lower bound for the left-hand side of the interaction Morawetz inequality (5.12). This lemma appears originally in the thesis of Visan.

Lemma A.2. *The following estimate holds:*

$$\||\nabla|^{-(\frac{d-3}{4})}f\|_{L_{t,x}^4}^4 \lesssim \||\nabla|^{-(\frac{d-3}{2})}|f|^2\|_{L_{t,x}^2}^2.$$

Proof. As these operators correspond to convolution with positive kernels (cf. (5.15) and (A.1)), it suffices to consider positive Schwartz functions f. The estimate will follow from the pointwise inequality

$$|S(|\nabla|^{-(\frac{d-3}{4})}f)(x)|^2 \lesssim \big(|\nabla|^{-(\frac{d-3}{2})}|f|^2\big)(x).$$

We work at an individual frequency, writing

$$\begin{aligned} P_N(|\nabla|^{-(\frac{d-3}{4})}f)(x) &= \int e^{-ix\cdot\xi}|\xi|^{-(\frac{d-3}{4})}\varphi(\tfrac{\xi}{N})\widehat{f}(\xi)\,d\xi \\ &=: N^{-(\frac{d-3}{4})}\int e^{-ix\cdot\xi}\tilde{\varphi}(\tfrac{\xi}{N})\widehat{f}(\xi)\,d\xi. \end{aligned}$$

Thus, we have

$$P_N(|\nabla|^{-(\frac{d-3}{4})}f)(x) \sim N^{\frac{3(d+1)}{4}}\int f(x-y)\mathcal{F}^{-1}\tilde{\varphi}(Ny)\,dy.$$

Using the rapid decay of $\mathcal{F}^{-1}\tilde{\varphi}$, we can estimate

$$|P_N(|\nabla|^{-(\frac{d-3}{4})}f)(x)| \lesssim N^{\frac{3(d+1)}{4}}\int_{|y|\lesssim N^{-1}} f(x-y)\,dy.$$

Thus, by Cauchy–Schwarz,

$$
\begin{aligned}
|S(|\nabla|^{-(\frac{d-3}{4})}f)(x)|^2 &\lesssim \sum_N N^{\frac{3(d+1)}{2}} \left| \int_{y|\lesssim N^{-1}} f(x-y)\,dy \right|^2 \\
&\lesssim \sum_N N^{\frac{d+3}{2}} \int_{|y|\lesssim N^{-1}} |f(x-y)|^2\,dy \\
&\lesssim \int |y|^{-(\frac{d+3}{2})} |f(x-y)|^2\,dy \lesssim \big(|\nabla|^{-\frac{d-3}{2}}|f|^2\big)(x),
\end{aligned}
$$

where in the last step we use

$$
\mathcal{F}^{-1}(|\xi|^{-(\frac{d-3}{2})}) = c|x|^{-(\frac{d+3}{2})} \quad \text{for some} \quad c > 0, \tag{A.1}
$$

which (like (5.15)) we leave as an exercise to the reader. The result follows. □

The following interpolation lemma also plays a role in deducing useful bounds in the interaction Morawetz inequality.

Lemma A.3. *Let $1 < r, r_1, r_2 < \infty$, $s_1, s_2 > 0$, and $\theta \in (0,1)$ satisfy*

$$
\tfrac{1}{r} = \tfrac{\theta}{r_1} + \tfrac{1-\theta}{r_2}, \quad -s_1\theta + (1-\theta)s_2 = 0.
$$

Then

$$
\|f\|_{L_x^r} \lesssim \||\nabla|^{-s_1} f\|_{L_x^{r_1}}^{\theta} \||\nabla|^{s_2} f\|_{L_x^{r_2}}^{1-\theta}.
$$

Proof. This follows, for example, by complex interpolation. However, we can also give a direct proof using the square function estimate and Hölder's inequality: writing $f_N := P_N f$ and using

$$
|f_N|^2 = (N^{-2s_1}|f_N|^2)^{\theta}(N^{2s_2}|f_N|^2)^{1-\theta},
$$

we have

$$
\begin{aligned}
\|f\|_{L_x^r} &\sim \left\| \Big(\sum_N (N^{-2s_1}|f_N|^2)^{\theta}(N^{2s_2}|f_N|^2)^{1-\theta} \Big)^{\frac{1}{2}} \right\|_{L_x^r} \\
&\lesssim \left\| \Big(\sum_N N^{-2s_1}|f_N|^2 \Big)^{\frac{\theta}{2}} \Big(\sum_N N^{2s_2}|f_N|^2 \Big)^{\frac{1-\theta}{2}} \right\|_{L_x^r} \\
&\lesssim \left\| \Big(\sum_N N^{-2s_1}|f_N|^2 \Big)^{\frac{1}{2}} \right\|_{L_x^{r_1}}^{\theta} \left\| \Big(\sum_N N^{2s_2}|f_N|^2 \Big)^{\frac{1}{2}} \right\|_{L_x^{r_2}}^{1-\theta}.
\end{aligned}
$$

The result follows. □

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