Math 185 - Spring 2015 - Homework 9 - Solution sketches

Problem 1. Show that

$$\int_0^1 x^s \, dx = \frac{1}{s+1}$$

for $s \in \mathbb{C}$ such that $\operatorname{Re} s > -1$.

Fix $\varepsilon > 0$. We have $x^s = e^{s \log x}$ for $x \in [\varepsilon, 1]$. Using Homework 3, Problem 6, we see that

$$F(s) = \int_\varepsilon^1 x^s \, dx$$

defines a holomorphic function for $(x, s) \in [\varepsilon, 1] \times \{s \in \mathbb{C} : \operatorname{Re} s > -1\}$. Furthermore since

$$F(s) = \frac{1}{s+1} - \frac{\varepsilon^{s+1}}{s+1}$$

whenever s is real, by the uniqueness theorem we deduce that $F(s) = \frac{1}{s+1} - \frac{\varepsilon^{s+1}}{s+1}$ for all $s \in \mathbb{C}$ such that $\operatorname{Re} s > -1$. Now send $\varepsilon \to 0$.

Problem 2. Suppose F is a meromorphic function on $\mathbb{C}$ and define $f(z) = \frac{F'(z)}{F(z)}$. Prove the following:

(i) If F has a zero of order n at z_0, then f has a simple pole at z_0 and $\operatorname{res}_{z_0} f = n$.

(ii) If F has a pole of order m at z_0, then f has a simple pole at z_0 and $\operatorname{res}_{z_0} f = -m$.

For (i), write $F(z) = (z - z_0)^n g(z)$ where g is a holomorphic function that is non-zero in a neighborhood of z_0. For (ii) write $F(z) = (z - z_0)^{-m} h(z)$ where h is a holomorphic function that is non-zero in a neighborhood of z_0. In both cases you can now compute that f has a simple pole at z_0 and find the residue.

Problem 3. For $s > 0$ define the **gamma function** by the convergent integral

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} \, dt.$$

(i) Show that the integral defining Γ defines a holomorphic function on $\{s \in \mathbb{C} : \operatorname{Re} s > 0\}$.

(ii) Show that Γ has a meromorphic continuation to all of $\mathbb{C}$. Determine its poles, their orders, and their residues.

For (i) the key is to see that the integral converges. To this end we split the integral into the regions where $t < 1$ and $t > 1$. For $t > 1$ the exponential factor e^{-t} guarantees convergence for any $s \in \mathbb{C}$, while for $t < 1$ we need $\operatorname{Re} s > 0$ to guarantee that t^{s-1} is absolutely integrable. (Note $|t^{s-1}| = t^{\operatorname{Re} s - 1}$.)

For (ii), we take $s \in \mathbb{C}$ with $\operatorname{Re} s > 0$ and split the integral as follows:

$$\int_0^\infty e^{-t} t^{s-1} \, dt = \int_0^1 e^{-t} t^{s-1} \, dt + \int_1^\infty e^{-t} t^{s-1} \, dt.$$

As the second integral converges (locally uniformly in s) for all $s \in \mathbb{C}$ we can see that this integral defines an entire function. For the first integral we write the power series for e^{-t} and integrate term by term to write this integral as

$$\sum_{n=0}^\infty \frac{(-1)^n}{n!(n+s)}.$$

One can then check that this sum actually defines a meromorphic function on $\mathbb{C}$, with poles at $s = 0, -1, -2, \ldots$ and with the residue at $s = -n$ equal to $(-1)^n / n!$.