Math 185 - Spring 2015 - Homework 8 - Solution sketches

Problem 1. Let $f : \mathbb{H} \to \mathbb{C}$ be holomorphic and satisfy $|f(z)| \le 1$ for $z \in \mathbb{H}$ and $f(i) = 0$. Show that

$$|f(z)| \le \left|\frac{z-i}{z+i}\right| \quad \text{for} \quad z \in \mathbb{H}.$$

Recall $g(z) = -i\frac{z+i}{z-i}$ is a biholmorphism from $\mathbb{D}$ to $\mathbb{H}$ and $g^{-1}(z) = \frac{i(z-i)}{z+i}$. Then $F = f \circ g$ is holomorphic, maps $\mathbb{D}$ to $\overline{\mathbb{D}}$, and $F(0) = f(g(0)) = f(i) = 0$. Thus by the Schwarz lemma we have $|F(z)| \le |z|$. Now $f(z) = F \circ g^{-1}(z) = F(\frac{i(z-i)}{z+i})$. Thus

$$|f(z)| \le |\tfrac{z-i}{z+i}|$$

as needed.

Problem 2. A **fixed point** of a function f is a point z such that $f(z) = z$.

(i) Show that if $f : \mathbb{D} \to \mathbb{D}$ is holomorphic and has two distinct fixed points, then $f(z) = z$ for $z \in \mathbb{D}$.

(ii) True or false: every holomorphic function $f : \mathbb{D} \to \mathbb{D}$ must have a fixed point. (Prove that your answer is correct.)

(i) Suppose that f has two distinct fixed points α and β.

First suppose that $\alpha = 0$. Then $f(0) = 0$ and so the Schwarz lemma implies that $|f(z)| \le |z|$ for $z \in \mathbb{D}$, with equality at some non-zero z implying that f is a rotation. But $f(\beta) = \beta$ implies $|f(\beta)| = |\beta|$, so that f is a rotation, that is, $f(z) = e^{i\theta}z$ for some θ. But $f(\beta) = \beta$ again implies that $\theta = 0$, that is, $f(z) = z$.

Now consider the general case, that is, α not necessarily zero. Then define $g = \psi_\alpha^{-1} \circ f \circ \psi_\alpha$, where ψ_α is as always the Blaschke factor. Then $g(0) = 0$ and since $f(\beta) = \beta$ we get $g(\psi_\alpha^{-1}(\beta)) = \psi_\alpha^{-1}(\beta)$. Thus the special case covered above implies that $g(z) = z$. This implies $f(\psi_\alpha(z)) = \psi_\alpha(z)$, which then gives $f(z) = z$.

(ii) False. Consider $f(z) = \frac{z+1}{2}$.

Problem 3. Let

$$A = \begin{pmatrix} -i & 1 \\ 1 & -i \end{pmatrix}, \quad M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Show that if $f_{A^{-1}} \circ g \circ f_A = e^{-2i\theta}$ then $g = f_M$.

Writing

$$B = \begin{pmatrix} e^{-2i\theta} & 0 \\ 0 & 1 \end{pmatrix}$$

we see that we need to show $f_A \circ f_B \circ f_{A^{-1}} = f_M$. To do this it suffices to show that $AB = \lambda MA$ for some non-zero λ. This follows from direct computation (matrix multiplication):

$$MA = \begin{pmatrix} -ie^{-i\theta} & e^{i\theta} \\ e^{-i\theta} & -ie^{i\theta} \end{pmatrix},$$

while

$$AB = \begin{pmatrix} -ie^{-2i\theta} & 1 \\ e^{-2i\theta} & -i \end{pmatrix} = e^{-i\theta}MA.$$

Problem 4. Suppose $\{f_n\}$ is a sequence of functions $f_n : \Omega \to \mathbb{C}$. Show that if $\{f_n\}$ is uniformly Cauchy on Ω, then $\{f_n\}$ converges uniformly on Ω.

For each $z \in \Omega$ we have that $f_n(z)$ converges to some limit, which we denote by $f(z)$. We will show f_n converges to f uniformly on Ω. To this end, we let $\varepsilon > 0$ and choose N so that $n, m \geq N$ implies $|f_n(z) - f_m(z)| < \varepsilon/2$ for any $z \in \Omega$. We claim that for any $z \in \Omega$ and any $n \geq N$ we have $|f_n(z) - f(z)| < \varepsilon$, which gives the result. Indeed, pick $z \in \Omega$ and $n \geq N$. Since we have convergence of the sequence $\{f_m(z)\}_m$ to $f(z)$ we may find $m \geq N$ so that $|f_m(z) - f(z)| < \varepsilon/2$. Thus

$$|f_n(z) - f(z)| \leq |f_n(z) - f_m(z)| + |f_m(z) - f(z)| < \varepsilon.$$

Problem 5. Let $\Omega \subset \mathbb{C}$ be open and $K \subset \Omega$ be compact. Show that there exists $r > 0$ such that for all $z \in K$ we have $B_r(z) \subset \Omega$.

Suppose not. Then for each n there exists $z_n \in K$ and $w_n \in \mathbb{C} \backslash \Omega$ such that $|z_n - w_n| < 1/n$. By compactness, z_n converges along a subsequence to some $z \in K$. Since $|z_n - w_n| < 1/n$, we deduce that w_n also converges to z along the same subsequence. But $\{w_n\}$ is a sequence in the closed set $\mathbb{C} \backslash \Omega$, so we must have $z \in \mathbb{C} \backslash \Omega$. Thus $z \in K \cap (\mathbb{C} \backslash \Omega) = \emptyset$, a contradiction.

Problem 6. Let $\{f_n\}$ be a sequence of functions $f_n : \Omega \to \mathbb{C}$ and $\{w_j\}_{j=1}^{\infty} \subset \Omega$. Suppose that for each $k \geq 1$ we have a subsequence $\{f_n^k\}$ of $\{f_n\}$ such that

$$\{f_n^{k+1}\} \quad \text{is a subsequence of} \quad \{f_n^k\}$$

and

$$\{f_n^k(w_j)\} \quad \text{converges for} \quad j = 1, \ldots, k.$$

Define the subsequence $\{g_n\}$ by $g_n = f_n^n$. Show that $\{g_n(w_j)\}$ converges for all j.

Fix some j. Then the tail $\{g_n\}_{n \geq j}$ is a subsequence of $\{f_n^j\}$. Since any subsequence of a convergent sequence converges, we find that the tail of $g_n(w_j)$ converges, which implies $g_n(w_j)$ converges.

Problem 7. Suppose that $\{K_\ell\}$ is a sequence of compact sets such that $K_\ell \subset K_{\ell+1}$ for each ℓ. Suppose that $\{f_n\}$ is a sequence of a functions and that for each $\ell \geq 1$ we have a subsequence $\{f_n^\ell\}$ of $\{f_n\}$ such that

$$\{f_n^{\ell+1}\} \quad \text{is a subsequence of} \quad \{f_n^\ell\}$$

and

$$\{f_n^\ell\} \quad \text{converges uniformly on } K_\ell.$$

Define the subsequence $\{g_n\}$ by $g_n = f_n^n$. Show that $\{g_n\}$ converges uniformly on each K_ℓ.

Fix ℓ. Then the tail $\{g_n\}_{n \geq \ell}$ is a subsequence of $\{f_n^\ell\}$, which converges uniformly on K^ℓ. Since any subsequence of a uniformly convergent sequence converges uniformly, we find that the tail of $\{g_n\}$ converges uniformly on K^ℓ. Thus g_n converges uniformly on K^ℓ.