Math 185 - Spring 2015 - Homework 7 - Solution sketches

Problem 1.

(i) Let (X,d) and $(Y,\tilde{d})$ be metric spaces and $f:X\to Y$. Show that f is continuous on X **if and only if**

$$\text{for all open} \quad U\subset Y, \quad f^{-1}(U) \quad \text{is open in } X, \quad (*)$$

where

$$f^{-1}(U) := \{x\in X : f(x)\in U\}.$$

(ii) Suppose $U,V\subset\mathbb{C}$ are open and non-empty and $f:U\to V$ is a biholomorphism. Show that the inverse $f^{-1}:V\to U$ is continuous.

(i) Suppose $(*)$ holds. Now let $x\in X$ and $\varepsilon>0$. As $B_\varepsilon(f(x))$ is open in Y, we have by $(*)$ that $f^{-1}(B_\varepsilon(f(x)))$ is open in X. As $x\in f^{-1}(B_\varepsilon(f(x)))$ we may therefore find $\delta>0$ so that $B_\delta(x)\subset f^{-1}(B_\varepsilon(f(x)))$. Thus $f(B_\delta(x))\subset B_\varepsilon(f(x))$. That is, $d(x,y)<\delta$ implies $\tilde{d}(f(x),f(y))<\varepsilon$, which gives continuity.

Next suppose f is continuous and suppose $U\subset Y$ is open. Suppose $x\in f^{-1}(U)$. As U is open, there exists $\varepsilon>0$ so that $B_\varepsilon(f(x))\subset U$. By continuity there exists $\delta>0$ so that $d(x,y)<\delta \implies \tilde{d}(f(x),f(y))<\varepsilon$. This means that $B_\delta(x)\subset f^{-1}(B_\varepsilon(f(x)))\subset f^{-1}(U)$, so that $f^{-1}(U)$ is open.

(ii) The open mapping theorem implies that f^{-1} satisfies the condition $(*)$, so that by part (i) f^{-1} is continuous.

Problem 2. Suppose $U,V\subset\mathbb{C}$ are open and biholomorphic.

(i) Show that $\mathrm{Aut}(U)$ and $\mathrm{Aut}(V)$ are isomorphic.

(ii) Show that if U is simply connected, then so is V.

(i) Let $F:U\to V$ be a biholomorphism. Then define $\varphi:\mathrm{Aut}(U)\to\mathrm{Aut}(V)$ by

$$\varphi(g)=F\circ g\circ F^{-1}.$$

One can check that φ is an isomorphism (it's the same argument we used to show that $\mathrm{Aut}(\mathbb{D})$ is isomorphic to $\mathrm{Aut}(\mathbb{H})$).

(ii) Let $\gamma_0(\cdot),\gamma_1(\cdot)$ be parametrized curves in V with common endpoints. Then $F^{-1}(\gamma_0)$ and $F^{-1}(\gamma_1)$ are curves in U with common endpoints. Since U is simply connected, there exists a function $\Gamma:[0,1]\times[0,1]\to U$ that witnesses the homotopy between $F^{-1}(\gamma_0)$ and $F^{-1}(\gamma_1)$. Then $F\circ\Gamma$ witnesses a homotopy between γ_0 and γ_1.

Problem 3. A (real) $n\times n$ matrix M is **orthogonal** if $MM^t=Id$, where ^{t} denotes transpose and Id is the $n\times n$ identity matrix. If M is orthogonal and $\det M=1$, we call M a **rotation**.

(i) Show that the set of orthogonal matrices forms a subgroup of $GL_2(\mathbb{R})$ and that the set of rotations forms a subgroup of $SL_2(\mathbb{R})$. (These groups are known as the orthogonal group, denoted $O(n)$, and the special orthogonal group, denoted $SO(n)$, respectively.)

(ii) Show that M is orthogonal **if and only if** $\langle Mv,Mw\rangle_{\mathbb{R}^n}=\langle v,w\rangle_{\mathbb{R}^n}$ for all $v,w\in\mathbb{R}^n$.

(iii) Show that if M is orthogonal then the angle between $v,w\in\mathbb{R}^n$ equals the angle between Mv and Mw.

(i) I'll give the argument for $GL_2(\mathbb{R})$, since the argument for $SL_2(\mathbb{R})$ is similar. We note that $MM^t=Id$ implies that M is invertible (since $M^t=M^{-1}$), and furthermore if $MM^t=NN^t=Id$ then

$$(MN)(MN)^t=MNN^tM^t=M(Id)M^t=MM^t=Id,$$

so that MN is also orthogonal. It follows that the set of orthogonal matrices is a subgroup.

(ii) If M is orthogonal, then

$$\langle Mv,Mw\rangle=\langle v,M^tMw\rangle=\langle v,w\rangle.$$

If instead we have $\langle Mv, Mw\rangle = \langle v, w\rangle$ for all v, w, then applying this to the standard basis vectors e_j and e_k we find

$$\delta_{jk} = \langle e_j, e_k\rangle = \langle Me_j, Me_k\rangle = \langle M^tMe_j, e_k\rangle = (M^tM)_{jk},$$

where $\delta_{jk} = 1$ if $j = k$ and 0 otherwise, and $(M^tM)_{jk}$ denotes the jk^{th} entry of M^tM. Thus $M^tM = Id$, which gives that M is orthogonal.

(iii) Recalling the definition of angle and noting that $\langle Mv, Mw\rangle = \langle v, w\rangle$ and (in particular) $\langle Mv, Mv\rangle = \langle v, v\rangle$, the result follows.

Problem 4. Show that $h(z) = -\frac{1}{2}(z + \frac{1}{z})$ is a holomorphic injective function from $\{z \in \mathbb{D} : \mathrm{Im}\, z > 0\}$ to $\mathbb{H}$.

That h is holomorphic is clear, since its domain does not contain 0. To see that it is injective, suppose $h(z) = h(w)$ for some $z, w \in \mathbb{D} \cap \mathbb{H}$. Some algebra then gives

$$z - w = \frac{z - w}{wz}.$$

If $z \neq w$, then this implies $wz = 1$, which means that z and w couldn't both belong to $\mathbb{D}$. Thus we must have $z = w$.

To see that h maps into $\mathbb{H}$, suppose $z = re^{i\theta} \in \mathbb{D} \cap \mathbb{H}$. Then $0 < r < 1$ and $\theta \in (0, \pi)$. We can then compute

$$\mathrm{Im} h(z) = \tfrac{1}{2}\sin\theta(\tfrac{1}{r} - r) > 0.$$

Problem 5. Consider the inverse of the stereographic projection map:

$$f : \mathbb{R}^2 \to \mathbb{R}^3, \quad f(x, y) = (\tfrac{x}{1+x^2+y^2}, \tfrac{y}{1+x^2+y^2}, \tfrac{x^2+y^2}{1+x^2+y^2}).$$

For $(x, y) \in \mathbb{R}^2$ let $M(x, y)$ denote the 3×2 matrix of partial derivatives of f at (x, y). Show that for all (x, y), one can write

$$[M(x, y)]^t M(x, y) = g(x, y)Id,$$

where ^{t} denotes transpose, Id is the 2×2 identity matrix, and $g : \mathbb{R}^2 \to \mathbb{R}$ is a strictly positive function.

We compute

$$M(x, y) = \frac{1}{(1 + x^2 + y^2)^2}\begin{pmatrix} 1 - x^2 + y^2 & -2xy \\ -2xy & 1 + x^2 - y^2 \\ 2x & 2y \end{pmatrix}.$$

Thus we find

$$[M(x, y)]^t M(x, y) = \frac{1 + x^4 + y^4 + 2x^2 + 2y^2}{(1 + x^2 + y^2)^4} Id,$$

which gives the result.

Problem 6. For each pair of sets U, V below, find a Möbius transformation taking U to V.

(i) $U = \{z \in \mathbb{C} : |z| > 1\}, \quad V = \{z \in \mathbb{C} : \mathrm{Re}\,(z) > 0\}$,

(ii) $U = \mathbb{D}, \quad V = \{z \in \mathbb{C} : \frac{\pi}{4} < \arg(z) < \frac{5\pi}{4}\}$.

(i) $f(z) = \frac{z-1}{z+1}$, (ii) $f(z) = e^{-i\pi/4}\frac{z+i}{z-i}$.

Problem 7. Let $M, N \in GL_2(\mathbb{C})$. Show that $f_{MN} = f_M \circ f_N$, where f_M is the Möbius transformation associated with M.

Let

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad N = \begin{pmatrix} e & f \\ g & h \end{pmatrix}.$$

Thus

$$MN = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}.$$

We now note

$$f_M(f_N(z)) = \frac{af_N(z) + b}{cf_N(z) + d} = \frac{a\frac{ez+f}{gz+h} + b}{c\frac{ez+f}{gz+h} + d} = \frac{(ae + bg)z + (af + bh)}{(ce + dg)z + (cf + dh)} = f_{MN}(z).$$