Math 185 - Spring 2015 - Homework 6 - Solution sketches

Problem 1.
(i) Construct a non-constant holomorphic function $f : \mathbb{D} \to \mathbb{C}$ such that f has infinitely many zeros inside $\mathbb{D}$.
(ii) Why does the existence of such a function not contradict the "uniqueness theorem"?

Let $f(z) = \sin(\frac{\pi}{1-z})$. Then $f : \mathbb{D} \to \mathbb{C}$ is holomorphic and non-constant, but if we define $z_k = 1 - \frac{1}{2k+1}$ for $k \geq 0$, then we have $z_k \in \mathbb{D}$ and $f(z_k) = 0$ (and these are all of the zeros of f in $\mathbb{D}$). This does not contradict the uniqueness theorem, because while z_k converges (to 1) the limit does not belong to $\mathbb{D}$.

Problem 2. Express the following functions as products:
(i) $f(z) = \sin(\pi z)$
(ii) $g(z) = e^z - 1$.

For (i) we first note that $\sin(\pi z)$ has order of growth equal to one (as we saw in class) and that its zeros are precisely at $k \in \mathbb{Z}$. Thus by the Hadamard factorization theorem we know that we must have

$$\sin(\pi z) = z\, e^{az+b} \prod_{k \in \mathbb{Z}\setminus\{0\}} (1 - \frac{z}{k}) e^{z/k} = z e^{az+b} \prod_{k=1}^{\infty} (1 - \frac{z^2}{k^2}).$$

We just need to determine a and b. Writing

$$\frac{\sin(\pi z)}{z} = e^{az+b} \prod_{k=1}^{\infty} (1 - \frac{z^2}{k^2})$$

and sending $z \to 0$, we deduce $e^b = \pi$. Thus

$$\sin(\pi z) = \pi z\, e^{az} \prod_{k=1}^{\infty} (1 - \frac{z^2}{k^2}).$$

Noting that $\sin(\pi z)$ is odd, πz is odd, and $\prod_{k=1}^{\infty} (1 - \frac{z^2}{k^2})$ is even, we deduce that e^{az} must be even. However, this means $e^{az} = e^{-az}$ for all $z \in \mathbb{C}$, or $e^{2az} = 1$ for all $z \in \mathbb{C}$. Thus $a = 0$. We conclude

$$\sin(\pi z) = \pi z \prod_{k=1}^{\infty} (1 - \frac{z^2}{k^2}).$$

For (ii) we note that $e^z - 1$ has order of growth equal to one and that its zeros are precisely at $2\pi i k$ for $k \in \mathbb{Z}$. Thus by the Hadamard factorization theorem we know that we must have

$$e^z - 1 = z\, e^{az+b} \prod_{k = \in \mathbb{Z}\setminus\{0\}}^{\infty} (1 - \frac{z}{2\pi i k}) e^{z/(2\pi i k)} = z\, e^{az+b} \prod_{k=1}^{\infty} (1 + \frac{z^2}{4\pi^2 k^2}).$$

We just need to determine a and b. Writing

$$\frac{e^z - 1}{z} = e^{az+b} \prod_{k=1}^{\infty} (1 + \frac{z^2}{4\pi^2 k^2})$$

and sending $z \to 0$ we deduce $e^b = 1$. Thus

$$e^z - 1 = z e^{az} \prod_{k=1}^{\infty} (1 + \frac{z^2}{4\pi^2 k^2}).$$

To figure out a we will evaluate both sides at $z = i\pi$. We find

$$-2 = i\pi e^{i\pi a} \prod_{k=1}^{\infty} (1 - \frac{1}{4k^2}).$$

To evaluate this infinite product we use the product formula for $\sin(\pi z)$. We find

$$\prod_{k=1}^{\infty} (1 - \frac{1}{4k^2}) = \frac{\sin(\pi/2)}{\pi/2} = \frac{2}{\pi}.$$

Plugging this into the above formula yields $ie^{ia\pi} = -1$, which implies $a = 1/2$. Thus

$$e^z - 1 = ze^{z/2} \prod_{k=1}^{\infty} (1 + \frac{z^2}{4\pi^2 k^2}).$$

Problem 3. Prove that

$$\frac{1}{1-z} = \prod_{k=0}^{\infty} (1 + z^{2^k}) \quad \text{for} \quad z \in \mathbb{D}.$$

We know that for $z \in \mathbb{D}$ we can write

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \dots$$

(in particular, that this series converges). On the other hand one can show (by induction, say) that

$$\prod_{k=0}^{N} (1 + z^{2^k}) = 1 + \cdots + z^{2^{N+1}-1}.$$

Sending $N \to \infty$ yields the result.

Problem 4. Suppose $f : \mathbb{C} \to \mathbb{C}$ is entire and has finite order of growth. Suppose that there exist distinct points $z_1, z_2 \in \mathbb{C}$ such that $f(z) \notin \{z_1, z_2\}$ for any $z \in \mathbb{C}$. Prove that f is constant.

Using Hadamard's factorization theorem we can write

$$f(z) - z_1 = e^{P_1(z)}, \quad f(z) - z_2 = e^{P_2(z)}$$

for some polynomials P_1, P_2. Thus

$$e^{P_1(z)} - e^{P_2(z)} = z_2 - z_1. \quad (*)$$

This implies

$$e^{P_1(z)} P_1'(z) = e^{P_2(z)} P_2'(z).$$

Thus P_1' and P_2' must have the same zeros, and hence be equal (since they are polynomials). This implies that $P_1(z) = P_2(z) + C$ for some $C \in \mathbb{C}$. But then $(*)$ implies

$$z_2 - z_1 = e^{P_2(z)+C} - e^{P_2(z)} = e^{P_2(z)}[e^C - 1].$$

As $z_2 \neq z_1$ we deduce $e^C - 1 \neq 0$, so that

$$e^{P_2(z)} = \frac{z_2 - z_1}{e^C - 1}.$$

This implies

$$f(z) = z_2 + \frac{z_2 - z_1}{e^C - 1},$$

so that f is constant, as was needed to show.

Problem 5. Suppose $f : \mathbb{C} \to \mathbb{C}$ is entire and has finite order of growth. Suppose that $f^{(n)}(z) \neq 0$ for any non-negative integer n and any $z \in \mathbb{C}$. Show that $f(z) = e^{az+b}$ for some $a, b \in \mathbb{C}$.

Since f has finite order of growth and no zeros, Hadamard's factorization theorem implies $f(z) = e^{P(z)}$ for some polynomial P. We now note that $f'(z) = e^{P(z)}P'(z)$ is nonzero for all z, which (by the fundamental theorem of algebra) implies that $P'(z) = b$ for some non-zero $b \in \mathbb{C}$. Thus $P(z) = az + b$ for some $a, b \in \mathbb{C}$.

Problem 6. How many solutions does the equation $e^z = z$ have in $\mathbb{C}$? Your options are: zero, finitely many, countably infinitely many, or uncountably many. (Prove that your answer is correct.)

We will equivalently count the number of zeros of the entire function $f(z) = e^z - z$. As f is not identically zero, the uniqueness theorem implies that f cannot have uncountably many zeros. However, we will see that f must have countably infinitely many zeros.

To see this, let's first show that f has at least one zero. If f had no zeros, then the Hadamard factorization theorem would imply that

$$e^z - z = e^{az+b}$$

for some $a, b \in \mathbb{C}$. Thus $e^{-(az+b)}(e^z - z) = 1$. However $e^{-(az+b)}(e^z - z)$ is clearly not constant for any choice of a, b, which gives a contradiction.

Now if f has finitely many zeros $a_1, \ldots, a_n \in \mathbb{C}\backslash$ then we apply a similar argument to

$$g(z) = \frac{f(z)}{(z - a_1)\cdots(z - a_n)}.$$

In particular we would have $g(z) = e^{az+b}$ for some $a, b \in \mathbb{C}$. However this implies that $e^{-(az+b)}(e^z - z)$ is a polynomial, which is not true for any $a, b \in \mathbb{C}$. This gives a contradiction.

We conclude that f has countably infinitely many zeros.

Problem 7.

(i) Find a sequence $\{a_n\} \subset \mathbb{C}$ such that $\sum_n a_n$ converges but $\prod_n (1 + a_n)$ diverges.
(ii) Find a sequence $\{a_n\} \subset \mathbb{C}$ such that $\prod_n (1 + a_n)$ converges but $\sum_n a_n$ diverges.

(i) Consider $\{a_n\} = \{i, -i, i/\sqrt{2}, -i/\sqrt{2}, \ldots\}$. The sum $\sum_n a_n$ converges (conditionally) by the alternating series test. But since

$$(1 + \frac{i}{\sqrt{n}})(1 - \frac{i}{\sqrt{n}}) = \frac{n+1}{n},$$

one can show that $\prod_{n=1}^{2k}(1 + a_n) = 2k + 1$, which diverges as $k \to \infty$.

(ii) There are some easier examples where the product is zero, but let's see one with a non-zero limit: consider a_n defined by $a_{2k} = \frac{1}{\sqrt{k}}$ and $a_{2k+1} = -\frac{1}{\sqrt{k}+1}$ for $k \geq 2$ (say). The sum diverges because if you group even and odd terms you end up with

$$\sum_k \frac{1}{\sqrt{k}(\sqrt{k}+1)}$$

which diverges. However for the product we note that

$$(1 + \frac{1}{\sqrt{k}})(1 - \frac{1}{\sqrt{k}+1}) = 1,$$

thus the partial products are either 1 or equal to

$$1 - \frac{1}{\sqrt{k}+1}$$

which converges to 1 as $k \to \infty$.