Math 185 - Spring 2015 - Homework 5 - Solution sketches

Problem 1. Let $z_0 \in \mathbb{C}$ and $R > 0$. Suppose that $f : B_R(z_0)\backslash\{z_0\}$ is holomorphic and that there exist $C > 0$ and $0 < \varepsilon < 1$ such that

$$|f(z)| \leq C|z - z_0|^{-1+\varepsilon} \quad \text{for all} \quad z \in B_R(z_0)\backslash\{z_0\}.$$

Show that the singularity of f at z_0 is removable, that is, there exists a unique holomorphic function $F : B_R(z_0) \to \mathbb{C}$ such that $F(z) = f(z)$ for $z \in B_R(z_0)\backslash\{z_0\}$.

We argue as in class. Without loss of generality we may assume f is holomorphic on some open set containing $B_R(z_0)$. We let

$$F(z) = \frac{1}{2\pi i}\int_{\partial B_R(z_0)} \frac{f(w)}{w - z}\,dw \quad \text{for} \quad z \in B_R(z_0).$$

We will show $F(z) = f(z)$ for $z \in B_R(z_0)\backslash\{z_0\}$. Using Cauchy's theorem and the Cauchy integral formula we find that for $\delta > 0$ small we have

$$F(z) = \frac{1}{2\pi i}\int_{\partial B_\delta(z_0)} \frac{f(w)}{w - z}\,dw + f(z).$$

Now for δ small we can estimate $|w - z| \geq \frac{1}{2}|z - z_0|$ for $w \in \partial B_\delta(z_0)$. Thus

$$\begin{aligned}\left|\frac{1}{2\pi i}\int_{\partial B_\delta(z_0)} \frac{f(w)}{w - z}\,dw\right| &\leq \frac{1}{2\pi}\int_{\partial B_\delta(z_0)} \frac{|f(w)|}{|w - z|}\,dw \\ &\leq \frac{1}{\pi|z - z_0|}\int_{\partial B_\delta(z_0)} C|w - z_0|^{-1+\varepsilon}\,dw \\ &\leq \frac{2\pi\delta C\delta^{-1+\varepsilon}}{\pi|z - z_0|} \\ &\leq \frac{2C\delta^{\varepsilon}}{|z - z_0|} \to 0 \quad \text{as} \quad \delta \to 0.\end{aligned}$$

Problem 2. Suppose $f : \mathbb{C} \to \mathbb{C}$ is entire, with $f(z) = \sum_{n=0}^{\infty} a_n z^n$.

(i) Show that if $a_n \neq 0$ for infinitely many n then f has an essential singularity at infinity.

(ii) Show that if f is injective then $f(z) = a_0 + a_1 z$ with $a_1 \neq 0$.

For (i) we consider the behavior of $F(z) = f(1/z)$ for z near zero. If F is bounded near zero or has a pole of order $k > 0$ near zero, then we can see that f is bounded near infinity or bounded by $|z|^k$ near infinity. Thus Liouville's theorem (in particular, the version from your homework) implies that f must be a polynomial. Thus f can only have an essential singularity at infinity.

For (ii) if f is a polynomial and injective, then (by fundamental theorem of algebra etc.) it has to be of the form $f(z) = a_0 + a_1 z$ with $a_1 \neq 0$. So it remains to show that f is a polynomial.

If f is not a polynomial then by part (i) it has an essential singularity at infinity. By Casorati–Weierstrass, we may find a sequence z_n such that $|z_n| \to \infty$ and $f(z_n) \to f(0)$. By the open mapping theorem there exists $r > 0$ such that $B_r(f(0)) \subset f(B_1(0))$. However, we may find n such that $|z_n| > 1$ and $f(z_n) \in B_r(f(0)) \subset f(B_1(0))$. This contradicts the injectivity of f.

Problem 3. Prove the following stronger version of Rouché's theorem.

Let $\Omega \subset \mathbb{C}$ be open and $\gamma \subset \Omega$ be a simple closed curve. Let $f, g : \Omega \to \mathbb{C}$ be holomorphic functions such that f has no zeros on γ and

$$|g(z)| < |f(z)| + |f(z) + g(z)| \quad \text{for all} \quad z \in \gamma.$$

Then f and $f + g$ have the same number of zeros in the interior of γ.

As in the proof of Rouchè's theorem from class, the key step is to prove that $|f(z) + tg(z)| > 0$ for $z \in \gamma$ and $t \in [0, 1]$. To do this we first use the triangle inequality to bound

$$|f(z) + tg(z)| \geq |f(z)| - t|g(z)|$$

as well as

$$|f(z) + tg(z)| \geq |f(z) + g(z) - (1 - t)g(z)| \geq |f(z) + g(z)| - (1 - t)|g(z)|.$$

Adding these two inequalities and dividing by 2 we find

$$|f(z) + tg(z)| \geq \tfrac{1}{2}\big[|f(z)| + |f(z) + g(z)| - |g(z)|\big] > 0,$$

as needed.

Problem 4. Use Rouchè's theorem to prove that any degree n polynomial has n zeros.

Let f be a polynomial of degree n. Write $f(z) = z^n + P(z)$ where P is a polynomial of degree $n-1$. Now, there exists $C > 0$ and $R > 0$ such that $|P(z)| \leq C|z|^{n-1}$ for $|z| > R$. We can also choose R possibly even larger so that $C|z|^{n-1} < |z|^n$ for $|z| > R$. Thus we have $|z^n| > |P(z)|$ for $z \in \partial B_R(0)$ and hence by Rouché's theorem we conclude that f has as many zeros as $z \mapsto z^n$ in $B_R(0)$. Since $z \mapsto z^n$ has n zeros (counting multiplicity) in $B_R(0)$, the result follows.

Problem 5. This problem provides an alternate proof of the maximum principle.

Let $\Omega = B_R(z_0)$ for some $z_0 \in \mathbb{C}$ and $R > 0$.

(i) Show that for any holomorphic function $f : \mathbb{C} \to \mathbb{C}$ we have

$$|f(z)| \leq \frac{R}{\inf_{w\in\partial\Omega} |z - w|} \sup_{w\in\partial\Omega} |f(w)| \quad \text{for all} \quad z \in \Omega.$$

(ii) Show that for any holomorphic function $g : \mathbb{C} \to \mathbb{C}$ we have

$$\sup_{z\in\Omega} |g(z)| \leq \sup_{w\in\partial\Omega} |g(w)|.$$

The inequality (i) follows from the Cauchy integral formula. For (ii) we apply (i) to the functions $f(z) = [g(z)]^n$ to find

$$|g(z)|^n \leq \frac{R}{\inf_{w\in\partial\Omega} |z - w|} \sup_{w\in\partial\Omega} |g(w)|^n.$$

Taking n^{th} roots we find

$$|g(z)| \leq \left(\frac{R}{\inf_{w\in\partial\Omega} |z - w|}\right)^{1/n} \sup_{w\in\partial\Omega} |g(w)|,$$

and sending n to infinity we get

$$|g(z)| \leq \sup_{w\in\partial\Omega} |g(w)|.$$

As this holds for all $z \in \Omega$, the result follows.