Math 185 - Spring 2015 - Homework 4 - Solution sketches

Problem 1. Compute $\int_0^\infty \sin(x^2)\,dx$ and $\int_0^\infty \cos(x^2)\,dx$.

We define $f(z) = e^{-z^2}$ (which is entire) and for $R > 0$ let γ_R be the union of $[0, R]$, $\{Re^{i\theta} : 0 \le \theta \le \pi/4\}$ and $\{re^{i\pi/4} : 0 \le r \le R\}$ with counter-clockwise orientation. We know that

$$\lim_{R\to\infty} \int_0^R e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}.$$

On the ray (oriented 'toward the origin') we have $f(re^{i\pi/4}) = e^{-ir^2} = \cos(r^2) - i\sin(r^2)$, so

$$\lim_{R\to\infty} \int_{\text{ray}} f(z)\,dz = e^{i\pi/4}\Big[-\int_0^\infty \cos(x^2)\,dx + i\int_0^\infty \sin(x^2)\,dx \Big].$$

For the arc we claim that we have

$$\lim_{R\to\infty} \int_{\text{arc}} f(z)\,dz = 0.$$

To see this we note

$$|f(z)| \le e^{-R^2\cos 2\theta}$$

for z on the arc. Thus we find the integral is bounded by

$$\int_0^{\pi/4} Re^{-R^2\cos 2\theta}\,d\theta.$$

We need to be a little careful to estimate this term since $\cos 2\theta \to 0$ as $\theta \to \pi/4$. So to estimate this, given $\varepsilon > 0$ we decompose $[0, \frac{\pi}{4}]$ into $[0, \frac{\pi}{4} - \frac{\varepsilon}{R}] \cup [\frac{\pi}{4} - \frac{\varepsilon}{R}, \frac{\pi}{4}]$. On the second interval we use that the length of the interval is $\frac{\varepsilon}{R}$ to get an estimate of ε for the integral. On the other interval we have a lower bound $\cos 2\theta \ge \frac{c\varepsilon}{R}$ for some $c > 0$ and we use that for any $\varepsilon > 0$

$$\lim_{R\to\infty} Re^{-c\varepsilon R} = 0.$$

Putting it all together we get that the integral over the arc vanishes in the limit.

Thus by Cauchy's theorem and all of the computations above we find

$$\int_0^\infty \cos(x^2)\,dx - i\int_0^\infty \sin(x^2)\,dx = e^{-i\pi/4}\frac{\sqrt{\pi}}{2}.$$

Taking real and imaginary parts we have

$$\int_0^\infty \cos(x^2)\,dx = \int_0^\infty \sin(x^2)\,dx = \frac{\sqrt{\pi}}{2\sqrt{2}}.$$

Problem 2. Compute $\int_0^\infty \frac{dx}{1+x^n}$ for all integers $n \ge 2$.

Consider $f(z) = \frac{1}{1+z^n}$. Let $R > 0$ and let γ_R be the contour consisting of $[0, R]$, the arc $\{Re^{i\theta} : 0 \le \theta \le \frac{2\pi}{n}\}$, and the ray $\{re^{\frac{2\pi i}{n}} : 0 \le r \le R\}$, oriented counter-clockwise. For all large R the function f has only one pole inside γ_R, namely the pole at $e^{\frac{\pi i}{n}}$.

Defining $I = \int_0^\infty \frac{dx}{1+x^n}$ and noting that $(e^{\frac{2\pi i}{n}})^n = 1$, we find that the integral along the ray (oriented 'toward the origin') is given by

$$-e^{\frac{2\pi i}{n}} \int_0^R \frac{dx}{1+x^n} \to -e^{\frac{2\pi i}{n}} I \quad \text{as } R \to \infty.$$

Moreover the integral over the arc can be bounded by

$$\frac{CR}{R^n}$$

for some $C > 0$ for large R and hence vanishes in the limit (since $n \geq 2$). Thus by the residue theorem we find

$$(1 - e^{\frac{2\pi i}{n}})I = 2\pi i \mathrm{Res}_{z_0} f, \quad z_0 = e^{\frac{\pi i}{n}},$$

and so

$$I = \frac{2\pi i}{1 - e^{\frac{2\pi i}{n}}} \mathrm{Res}_{z_0}(f).$$

It remains to calculate the residue. To this end we factor

$$\frac{1}{1+z^n} = \prod_{k=0}^{n-1} \frac{1}{z - e^{\frac{i(2k+1)\pi}{n}}},$$

so that

$$\mathrm{Res}_{z_0}(f) = \prod_{k=1}^{n-1} \frac{1}{e^{\frac{\pi i}{n}} - e^{\frac{i(2k+1)\pi}{n}}} = \frac{1}{e^{\frac{(n-1)i\pi}{n}}} \prod_{k=1}^{n-1} \frac{1}{1 - e^{\frac{2ki\pi}{n}}}.$$

Thus

$$I = \frac{2\pi i}{e^{\frac{(n-1)i\pi}{n}}} \frac{1}{1 - e^{\frac{2\pi i}{n}}} \prod_{k=1}^{n-1} \frac{1}{1 - e^{\frac{2ki\pi}{n}}}. \quad (*)$$

We now claim that

$$\prod_{k=1}^{n-1} \frac{1}{1 - e^{\frac{2ki\pi}{n}}} = \frac{1}{n}. \quad (**)$$

To see this, we first factor

$$z^n - 1 = (z-1)(z^{n-1} + z^{n-2} + \cdots + z + 1).$$

However we can also compute the n^{th} roots of 1 and factor

$$z^n - 1 = (z-1) \prod_{k=1}^{n-1} (z - e^{i\frac{2ki\pi}{n}}).$$

Equating the two formuals for $z^n - 1$ we find

$$\prod_{k=1}^{n-1} (z - e^{i\frac{2ki\pi}{n}}) = z^{n-1} + z^{n-2} + \cdots + z + 1,$$

and evaluating at $z = 1$ yields the result (since there are n terms on the right-hand side).

Thus continuing from $(*)$ and using Euler's formula we find

$$I = \frac{2\pi i}{n} \frac{1}{e^{i\pi}[e^{-\frac{i\pi}{n}} - e^{\frac{i\pi}{n}}]} = \frac{\pi}{n \sin(\frac{\pi}{n})}.$$

Problem 3. Compute $\int_0^{2\pi} \frac{d\theta}{2+\cos^2\theta}$.

If we write $z = z(\theta) = e^{i\theta}$ and $z'(\theta) = iz(\theta)$ for $\theta \in [0, 2\pi]$ then we can recognize this as an integral over $\partial B_1(0)$. In this case $e^{-i\theta} = \frac{1}{z(\theta)}$ and so $\cos\theta = \frac{1}{2}(z(\theta) + \frac{1}{z(\theta)})$. Then we have

$$\begin{aligned} \int_0^{2\pi} \frac{d\theta}{2 + \cos^2(\theta)} &= \int_0^{2\pi} \frac{z'(\theta)\, d\theta}{iz(\theta)(2 + \frac{1}{4}(z(\theta) + \frac{1}{z(\theta)})^2)} \\ &= 2\pi \left(\frac{1}{2\pi i} \int_{\partial B_1(0)} \frac{4z\, dz}{z^4 + 10z^2 + 1} \right). \end{aligned}$$

So we just need to find the poles of the function

$$f(z) = \frac{4z}{z^4 + 10z^2 + 1}$$

inside the unit circle and compute the residues at each pole. One computes that there are simple poles at $i\sqrt{5 - 2\sqrt{6}}$ and $-i\sqrt{5 - 2\sqrt{6}}$ and the residue at each pole equals $\frac{1}{2\sqrt{6}}$. Thus by the residue theorem

$$\int_0^{2\pi} \frac{d\theta}{2 + \cos^2(\theta)} = 2\pi \cdot 2 \cdot \frac{1}{2\sqrt{6}} = \frac{2\pi}{\sqrt{6}}.$$

Problem 4. Compute $\int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)(4+x^2)}\, dx$.

Let γ_R denote the union of the line segment $[-R, R]$ and the semicircle $\{Re^{i\theta} : \theta \in [0, \pi]\}$. We first note that

$$\int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)(4+x^2)}\, dx = \lim_{R\to\infty} \int_{\gamma_R} \frac{z^2}{(1+z^2)(4+z^2)}.$$

Indeed, this follows from the fact that for R sufficiently large, the integral over the semicircle is bounded by $\frac{CR^3}{R^4}$ for some $C > 0$, and hence vanishes in the limit as $R \to \infty$. By the residue theorem, we can compute the limit as $2\pi i$ times the sum of the residues contained in the upper half plane.

We can compute there are two poles in the upper half plane, namely i and $2i$, and that the residues at these poles are $\frac{1}{6}i$ and $-\frac{1}{3}i$, respectively. Thus

$$\int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)(4+x^2)}\, dx = 2\pi\left(\frac{1}{3} - \frac{1}{6}\right) = \frac{\pi}{3}.$$

Problem 5. Compute $\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2}\, dx$ for $a > 0$.

We use the curve γ_R from Problem 4. We let $f(z) = \frac{e^{iz}}{z^2+\alpha^2}$. Then

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2 + a^2}\, dx = \lim_{R\to\infty} \int_{\gamma_R} \frac{e^{iz}}{z^2 + a^2}\, dx.$$

The integral over the semicircle goes to zero in the limit as $R \to \infty$, since for large R (much larger than a, say) the integral is bounded by $\frac{CR}{R^2}$ for some $C > 0$. Furthermore writing $e^{ix} = \cos x + i \sin x$ and noting that

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{\sin x}{x^2 + a^2}\, dx = 0$$

(since sin is an odd function), we find

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2}\, dx$$

is equal to $2\pi i$ times the residues of the poles of $f(z)$ in the upper half plane. We now compute that f has a single pole at ia with residue equal to $\frac{e^{-a}}{2ia}$. Thus

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2}\, dx = \frac{\pi e^{-a}}{a}.$$

Problem 6. Compute $\int_{-\infty}^{\infty} \frac{x^3 \sin x}{x^4+16}\, dx$.

Consider the function $f(z) = \frac{e^{iz}z^3}{z^4+16}$ over the contour γ_R from the previous two problems. We can estimate the contribution of the integral over the semicircle by

$$\frac{CR^3}{R^4}\int_0^\pi Re^{-R\sin\theta}\,d\theta$$

for large R and for some $C > 0$. We now note that since $\sin\theta \geq c\theta$ for some small $c > 0$ for all $\theta \in [0, \pi/2]$ we can write

$$\int_0^\pi Re^{-R\sin\theta}\,d\theta = 2R\int_0^{\pi/2} e^{-R\sin\theta}\,d\theta \leq 2R\int_0^{\pi/2} e^{-cR\theta}\,d\theta \leq C$$

for some $C > 0$ and all $R > 0$. Thus the integral over the semicircle goes to zero as $R \to \infty$. So we find

$$\int_{-\infty}^{\infty} \frac{x^3\sin x}{x^4+16}\,dx$$

is the imaginary part of $2\pi i$ times the sum of the residues of f in the upper half plane.

We compute that f has simple poles at $\sqrt{2}(1+i)$ and $\sqrt{2}(-1+i)$, with residues equal to $\frac{1}{4}e^{-\sqrt{2}+i\sqrt{2}}$ and $\frac{1}{4}e^{-\sqrt{2}-i\sqrt{2}}$. Thus

$$\int_{-\infty}^{\infty} \frac{x^3\sin x}{x^4+16}\,dx = \pi e^{-\sqrt{2}}\cos\sqrt{2}.$$

Problem 7. Compute $\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{n+1}}$ for all integers $n \geq 0$.

Once again take γ_R as above. The function $f(z) = \frac{1}{(1+z^2)^{n+1}}$ has one pole at i of order $n+1$ in the upper half plane. The integral over the semicircle is bounded by

$$\frac{CR}{R^{2(n+1)}}$$

for some $C > 0$ and hence vanishes in the limit as $R \to \infty$. Thus we find

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{n+1}} = 2\pi i \mathrm{Res}_i f.$$

We need to compute

$$\mathrm{Res}_i f = \lim_{z\to i} \frac{1}{n!}\left(\frac{d}{dz}\right)^n[(z-i)^{n+1}f(z)] = \lim_{z\to i} \frac{1}{n!}\left(\frac{d}{dz}\right)^n[(z+i)^{-(n+1)}] = \frac{(-1)^n(n+1)\cdots(2n)}{n!(2i)^{2n+1}}.$$

Thus the integral equals

$$\begin{aligned}\frac{2\pi i(-1)^n(n+1)\cdots(2n)}{n!2(i^2)^n2^{2n}i} &= \frac{(2n!)}{n!n!2^{2n}}\pi = \frac{1\cdots 2n}{2\cdots 2}\frac{\pi}{2^n n!n!} \\ &= \frac{[1\cdot 3\cdot 5\cdots(2n-1)]n!}{2^n n!n!}\pi = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}\pi.\end{aligned}$$