Math 185 - Spring 2015 - Homework 3 - Solution sketches

Problem 1. Show that the relation "is homotopic to" is an equivalence relation. That is,
(i) any curve is homotopic to itself,
(ii) if γ_0 is homotopic to γ_1, then γ_1 is homotopic to γ_0,
(ii) if γ_0 is homotopic to γ_1 and γ_1 is homotopic to γ_2, then γ_0 is homotopic to γ_2.

(i) If $t \mapsto \gamma(t)$ parametrizes a curve then $(s,t) \mapsto \gamma(t)$ is is a homotopy between the curve and itself.

(ii) if $(s,t) \mapsto \gamma(s,t)$ (for $s,t \in [0,1]$) witnesses the homotopy between γ_0 and γ_1, then $(s,t) \mapsto \gamma(1-s,t)$ witnesses the homotopy between γ_1 and γ_0.

(iii) if $(s,t) \mapsto \gamma(s,t)$ (for $s,t \in [0,1]$) witnesses the homotopy between γ_0 and γ_1 and $(s,t) \mapsto \tilde{\gamma}(s,t)$ (for $s,t \in [0,1]$) witnesses the homotopy between γ_1 and γ_2, then the function

$$(s,t) \mapsto \begin{cases} \gamma(2s,t) & s \in [0,1/2] \\ \tilde{\gamma}(2s-1,t) & s \in (1/2,1] \end{cases}$$

witnesses the homotopy between γ_0 and γ_2.

Problem 2. Let $r, R > 0$ and $z_0, z_1 \in \mathbb{C}$. Construct a **continuous** $F : [0,1] \times [0,1] \to \mathbb{C}$ such that

- the function $t \mapsto F(0,t)$ is a parametrization of $\partial B_r(z_0)$,
- the function $t \mapsto F(1,t)$ is a parametrization of $\partial B_R(z_1)$,
- for each $s \in (0,1)$ the function $t \mapsto F(s,t)$ parametrizes a closed curve in $\mathbb{C}$.

One can check that $F(s,t) = (1-s)z_0 + sz_1 + [(1-s)r + sR]e^{2\pi i\theta}$ satisfies all the requirements.

Problem 3. Prove this stronger version of the Cauchy integral formula: let $\Omega \subset \mathbb{C}$ be open and $f : \Omega \to \mathbb{C}$ be holomorphic. If $z_0 \in \Omega$ and B is any ball containing z_0 such that $\overline{B} \subset \Omega$, then

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial B} \frac{f(z)}{z - z_0}\, dz.$$

We know that $f(z_0) = \frac{1}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(z)}{z-z_0}\, dz$ for all sufficiently small $r > 0$ by the Cauchy integral formula proved in class. In particular we can take r small enough that $\overline{B_r(z_0)} \subset B$. Now we argue as in class, joining ∂B to $\partial B_r(z_0)$ by two vertical lines and applying Cauchy's theorem to the appropriate closed curves. Rearranging the results yields that the integrals over the two circles are the same, which proves the result.

Problem 4. Let $K \subset \mathbb{C}$ be compact and $f : K \to \mathbb{C}$ be continuous. Suppose that $f(z) \neq 0$ for all $z \in K$. Show that

$$\text{there exists} \quad \delta > 0 \quad \text{such that} \quad \text{for all} \quad z \in K \quad |f(z)| \geq \delta.$$

Suppose not. Then for all n there exists $z_n \in K$ such that $|f(z_n)| \leq \frac{1}{n}$. By compactness, z_n converges along a subsequence to some $z \in K$. By continuity $f(z) = 0$, a contradiction.

Problem 5. Let $\Omega \subset \mathbb{C}$ be open and suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence of holomorphic functions on Ω that converge uniformly to $f : \Omega \to \mathbb{C}$. Show that for $\delta > 0$ we have that $f_n' \to f'$ uniformly on the set

$$K_\delta := \{z \in \Omega : \overline{B_\delta(z)} \subset \Omega\}.$$

Define the sequence of holomorphic functions $F_n = f_n - f$ and let $\varepsilon > 0$. By uniform convergence we may find N such that $|F_n(z)| \leq \delta\varepsilon$ for all $z \in \Omega$ and all $n \geq N$. Now for all $z \in K_\delta$ we have $\overline{B_\delta(z)} \subset \Omega$ so that we can apply the Cauchy inequalities on $\partial B_\delta(z)$. We find that for $n \geq N$ and

$z \in K_\delta$,

$$|F_n'(z)| \le \frac{1}{\delta} \sup_{w \in \partial B_\delta(z)} |F_n(w)| < \frac{\delta\varepsilon}{\delta} = \varepsilon.$$

We conclude $F_n' \to 0$ uniformly on K_δ; that is, $f_n' \to f'$ uniformly on K_δ.

Problem 6. Let $\Omega \subset \mathbb{C}$ be open. Suppose $F : [0,1] \times \Omega \to \mathbb{C}$ is continuous and satisfies

$$\text{for all} \quad s \in [0,1] \quad \text{the function} \quad z \mapsto F(s,z) \quad \text{is holomorphic on } \Omega.$$

Show that the function $f : \Omega \to \mathbb{C}$ defined by $f(z) = \int_0^1 F(s,z)\,ds$ is holomorphic on Ω.

For any n define the points $s_k = \frac{k}{n}$ for $k = 0, \ldots, n$ and the intervals $I_k = [s_{k-1}, s_k]$ for $k = 1, \ldots, n$.

Consider the Riemann sum

$$f_n(z) := \sum_{k=1}^n \tfrac{1}{n} F(s_k, z) = \sum_{k=1}^n \int_{I_k} F(s_k, z)\,ds.$$

Note that as each f_n is the finite sum of holomorphic functions we have that each f_n is holomorphic. We will show that $f_n \to f$ locally uniformly, which implies that f is holomorphic.

To this end let $K \subset \Omega$ be compact and let $\varepsilon > 0$. As F is continuous and $[0,1] \times K$ is compact, we have that F is uniformly continuous on $[0,1] \times K$. Thus there exists $\delta > 0$ such that

$$\sup_{z \in K} |F(s,z) - F(\tilde{s}, z)| < \varepsilon \quad \text{whenever} \quad |s - \tilde{s}| < \delta.$$

Choosing $n > \frac{1}{\delta}$ (so that $|s - s_k| < \delta$ for $s \in I_k$) we find that for any $z \in K$ we have

$$\begin{aligned} |f(z) - f_n(z)| &= \left| \int_0^1 F(s,z)\,ds - \sum_{k=1}^n \int_{I_k} F(s_k, z)\,ds \right| \\ &= \left| \sum_{k=1}^n \int_{I_k} [F(s,z) - F(s_k,z)]\,ds \right| \\ &\le \sum_{k=1}^n \int_{I_k} |F(s,z) - F(s_k,z)|\,ds \\ &< \sum_{k=1}^n \int_{I_k} \varepsilon\,ds \le \int_0^1 \varepsilon\,ds < \varepsilon, \end{aligned}$$

which gives the result.

Problem 7. Can every continuous function on the set $\{z \in \mathbb{C} : |z| \le 1\}$ be approximated uniformly by polynomials? If so, prove it. If not, give a counterexample.

No, because the uniform limit of holomorphic functions is holomorphic, but not every continuous function is holomorphic (e.g. $f(z) = \bar{z}$).

Problem 8. Suppose $f : \mathbb{C} \to \mathbb{C}$ is entire and satisfies $|f(z)| \le C(1 + |z|)^n$ for some $C > 0$ and some integer n (for all $z \in \mathbb{C}$). Show that f is a polynomial of degree at most n.

By the Cauchy inequalities we have that for any $R > 0$ and $k > 0$,

$$|f^{(n+k)}(0)| \le \frac{1}{R^{n+k}} \sup_{|z|=R} |f(z)| \le \frac{C(1+|R|)^n}{R^{n+k}} \to 0 \quad \text{as} \quad R \to \infty.$$

Thus $f^{(n+k)}(0) = 0$ for all $k > 0$. Examining the power series expansion for f at zero (valid on all of $\mathbb{C}$), say $f(z) = \sum a_k z^k$, and recalling that $a_k = \frac{f^{(k)}(0)}{k!}$ we conclude that f is a polynomial of degree at most n.

Problem 9. Suppose $f : \mathbb{C} \to \mathbb{C}$ is entire.

(i) Show that if $f(z) = 0$ for uncountably many $z \in \mathbb{C}$ then $f \equiv 0$.

(ii) Suppose that for each $z_0 \in \mathbb{C}$ at least one coefficient in the power series expansion at z_0 is zero. Prove that f is a polynomial.

(i) As any uncountable set has a limit point, this result follows from the "uniqueness theorem" from class.

(ii) As there are uncountably many points in $\mathbb{C}$ but only countably many positive integers, we conclude that for some n the coefficient a_n in the power series expansion of f is zero for uncountably many $z \in \mathbb{C}$. This means that the function $z \mapsto f^{(n)}(z)$ is zero for uncountably many $z \in \mathbb{C}$. By part (i) this means that $f^{(n)} \equiv 0$, which implies that f is a polynomial.