Math 185 - Spring 2015 - Homework 2 - Solution sketches

Problem 1. Show that if f is holomorphic at $z \in \mathbb{C}$ then f is continuous at z.

For any $w \neq z$ we have

$$f(z) - f(w) = (z - w)\frac{f(z) - f(w)}{z - w} \to 0 \cdot f'(z) = 0 \quad \text{as} \quad w \to z.$$

Problem 2. Let $\Omega \subset \mathbb{C}$ be open. Show that Ω is connected if and only if it is path connected.

Suppose Ω is connected. To show Ω is path connected, it suffices to show that for any $z_0 \in \Omega$, the set

$$A := \{z \in \Omega : \text{there exists a curve in } \Omega \text{ joining } z_0 \text{ to } z\}$$

is equal to Ω. So fix $z_0 \in \Omega$ and define A as above. As Ω is connected, to prove $A = \Omega$ we can show A is non-empty, closed in Ω, and open in Ω. The set A is clearly non-empty, since $z_0 \in A$. Next, suppose $\{z_n\} \subset A$ converges to some $z \in \Omega$. Since Ω is open, we may find $r > 0$ such that $B_r(z) \subset \Omega$. As $z_n \to z$ we may find $z_n \in B_r(z)$. Since $z_n \in A$ we may join z_0 to z_n with a curve in Ω. We can then concatenate this curve with the line segment joining z_n to z, thus giving a curve in Ω joining z_0 to z. In particular $z \in A$, so that A is closed in Ω. Finally suppose $z \in A$. Then since Ω is open we can find some $r > 0$ such that $B_r(z) \subset \Omega$. We can now find curves in Ω joining z_0 to any $w \in B_r(z)$ by first joining z_0 to z (possible since $z \in A$) and then following the line segment from z to w. In particular $B_r(z) \subset A$, so that A is open in Ω.

Now suppose Ω is path connected. To show Ω is connected, we argue as follows: suppose $A \subset \Omega$ is non-empty, open in Ω and closed in Ω. We will show $A = \Omega$. If not, we can find $z \in \Omega \backslash A$. Now we take $w \in A$. As Ω is path connected we may find a curve joining w to z. Let $\gamma : [0, 1] \to \Omega$ be a parametrization of this curve and define $t_* = \sup\{t \in [0, 1] : \gamma(s) \in A \text{ for } 0 \leq s < t\}$. Now by continuity of γ and the fact that A is closed, we have $\gamma(t_*) \in A$. This also shows $t_* < 1$. However, since A is open we can then find $t \in (t_*, 1)$ such that $\gamma(s) \in A$ for all $0 \leq s < t$, contradicting the definition of t_*.

Problem 3. Let $\Omega \subset \mathbb{C}$ be open and connected and $f : \Omega \to \mathbb{C}$ be holomorphic. Show that if $f'(z) = 0$ for all $z \in \Omega$ then f is constant.

Since f is a primitive for f', we find that

$$f(\beta) - f(\alpha) = \int_\gamma f'(z)\, dz = \int_\gamma 0\, dz = 0$$

for any $\alpha, \beta \in \Omega$, where γ is any curve joining α to β. (Here we have used that Ω is open and connected, and hence path connected.)

Problem 4. Suppose $\Omega \subset \mathbb{C}$ is open and connected and $f : \Omega \to \mathbb{C}$ is continuous. Show that if F and $\tilde{F}$ are both primitives for f in Ω then the function $F - \tilde{F}$ is constant.

As $(F - \tilde{F})' = f - f = 0$, Problem 3 implies $F - \tilde{F}$ is constant.

Problem 5.

- Show that $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is simply connected.
- Find an open connected subset of $\mathbb{C}$ that is **not** simply connected. (Explain why your example meets all of the stated requirements.)

(i) Suppose γ_0 and γ_1 are two piecewise-smooth curves in $\mathbb{D}$ with common endpoints. Let $t \mapsto \gamma_0(t)$ and $t \mapsto \gamma_1(t)$ be parametrizations of γ_0, γ_1 for $t \in [0,1]$. Define $\gamma : [0,1] \times [0,1] \to \mathbb{C}$ by

$$\gamma(s,t) = (1-s)\gamma_0(t) + s\gamma_1(t).$$

The convexity of $\mathbb{D}$ implies $\gamma(s,t) \in \mathbb{D}$ for each $(s,t) \in [0,1] \times [0,1]$. The function γ is continuous since it is the sum/product of continuous functions. Moreover $\gamma(0,t) = \gamma_0(t)$ and $\gamma(1,t) = \gamma_1(t)$, and at each $s \in (0,1)$ the function $t \mapsto \gamma(s,t)$ parametrizes a curve with the same endpoints as γ_0, γ_1 that is piecewise-smooth.

(ii) I claim that $\Omega := \mathbb{D} \backslash \{0\}$ is open, connected, but not simply connected. It is clear that it is open, since it is the intersection of two open sets. It is also path connected. Indeed, one can first join any two points with a straight line—if that straight line happens to pass through the origin, one can easily modify the curve to miss the origin (for example, by following along part of the boundary of a small ball around zero). As it is open and path connected, it must be connected by Problem 2.

However, Ω is not simply connected. Indeed, suppose toward a contradiction that it were simply connected. Then the function $f(z) = \frac{1}{z}$ is holomorphic in Ω, but

$$\int_{\partial B_{1/2}(0)} f(z)\, dz = 2\pi i \neq 0,$$

contradicting Cauchy's theorem.

Problem 6. Let γ be a circle with positive orientation.

- Suppose γ is centered at the origin. Evaluate the integrals

$$\int_\gamma z^n\, dz \quad \text{for} \quad n \in \mathbb{Z}. \quad (*)$$

- Suppose γ does not contain the origin. Evaluate the integrals $(*)$.

(i) For $n \neq -1$ we get the value 0. For $n = -1$ we get $2\pi i$.
(ii) For all n we get 0.

Problem 7. Let $\Omega \subset \mathbb{C}$ be open and $f : \Omega \to \mathbb{C}$ be holomorphic. Further assume that f' is continuous. Use **Green's theorem** to show that

$$\int_{\partial T} f(z)\, dz = 0$$

for any triangle $T \subset \Omega$.

We write $f = u + iv$ and view this as a line integral in $\mathbb{R}^2$:

$$\int_{\partial T} f(z)\, dz = \int_{\partial T} u\, dx - v\, dy + i \int_{\partial T} v\, dx + u\, dy.$$

Thus by Green's theorem we have

$$\int_{\partial T} f(z)\, dz = \int_T -\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\, dx\, dy + i \int_T \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\, dx\, dy.$$

However by the Cauchy–Riemann equations both of these integrals are zero.