Math 185 - Spring 2015 - Homework 1 - Solution sketches

Hard copy due: Tuesday, February 3 at 11am.

Notation. \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}, \quad \partial \mathbb{D} = \{ z \in \mathbb{C} : |z| = 1 \}. \)

**Problem 1.** For all \( z \in \mathbb{C}\setminus\{0\} \) there exists a unique \( w \in \mathbb{C}\setminus\{0\} \) such that \( zw = 1 \), which we denote by \( \frac{1}{z} \) or \( z^{-1} \). Given \( z = x + iy \in \mathbb{C}\setminus\{0\} \), compute the real and imaginary parts of \( z^{-1} \).

If \( z = x + iy \) then \( z^{-1} = \frac{x}{x^2 + y^2} + i\frac{-y}{x^2 + y^2} \).

**Problem 2.** Describe the following sets in \( \mathbb{C} \) geometrically and draw a picture of each.
- \( \{ z \in \mathbb{C} : |z - a| = |z - b| \} \), where \( a, b \in \mathbb{C} \),
- \( \{ z \in \mathbb{C} : \text{Re} (z) > 0 \} \),
- \( \{ z \in \mathbb{C} : \text{Re} (az + b) > 0 \} \), where \( a, b \in \mathbb{C} \),
- \( \{ z \in \mathbb{C} : |z| = \text{Re} (z) + 1 \} \).

I’ll describe the sets as subsets of the plane. The first set is the line of points equidistant to \((a_1, a_2)\) and \((b_1, b_2)\). The second set is the right-half plane. The third set is the set of points such that \( a_2y < a_1x + b_1 \) (also a half plane). The fourth set is the parabola \( y^2 = 2x + 1 \).

**Problem 3.** Define \( f : \mathbb{C} \to \mathbb{C} \) by \( f(z) = \bar{z} \). Use the definition of the derivative to show that \( f \) is not holomorphic at any point.

Note \( f(z+h) - f(z) = \frac{h}{\bar{h}} \). If we send \( h \to 0 \) along the real axis we get \( 1 \). If we send \( h \to 0 \) along the imaginary axis we get \(-1 \). Thus the limit as \( h \to 0 \) does not exist.

**Problem 4.** Fix \( w \in \mathbb{D} \) and define the Blaschke factor
\[
F(z) = \frac{w - z}{1 - \bar{w}z} \quad \text{for} \quad z \in \mathbb{D}.
\]

Show the following:
- \( F : \mathbb{D} \to \mathbb{D} \), and \( F : \partial \mathbb{D} \to \partial \mathbb{D} \),
- \( F \) is a bijection on \( \mathbb{D} \),
- \( F \) is holomorphic on \( \mathbb{D} \).

Writing \( z = re^{i\theta} \) we find
\[
F(z) = e^{i\theta} \frac{w_0 - r}{1 - \bar{w}_0 r}, \quad \text{where} \quad w_0 = e^{-i\theta} w.
\]

As \( |e^{\pm i\theta}| = 1 \), we see that without loss of generality we may assume \( z \in \mathbb{R} \).

Now for \( z \in \mathbb{R} \) the statement \(|F(z)|^2 \leq 1 \) can be seen to be equivalent to
\[
|w|^2 + z^2 - 2z\text{Re} (w) \leq 1 + z^2 |w|^2 - 2z\text{Re} (w),
\]
or equivalently
\[
|w|^2 (1 - z^2) \leq 1 - z^2.
\]
This inequality holds (strictly) when \( |z| < 1 \), while equality holds if \( |z| = 1 \).

To see that \( F \) is a bijection we note that it is its own inverse.

To see that \( F \) is holomorphic we note that it is a quotient of holomorphic functions and the denominator is never zero, since \( |\bar{w}z| = |w| \cdot |z| < 1 \).

**Problem 5.** Let \( f : \mathbb{C} \to \mathbb{C} \) and define \( u, v : \mathbb{R}^2 \to \mathbb{R} \) by
\[
u(x, y) = \text{Re} [f(x + iy)] , \quad v(x, y) = \text{Im} [f(x + iy)].
\]
Suppose \( f \) is holomorphic at some \( z_0 = x_0 + iy_0 \in \mathbb{C} \).
Use the definition of the derivative to show that
\[ f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial u}{\partial y}(x_0, y_0) \quad \text{and} \quad f'(z_0) = -i\frac{\partial u}{\partial y}(x_0, y_0) + \frac{\partial u}{\partial x}(x_0, y_0). \quad (*) \]

Use (*) to derive the Cauchy–Riemann equations.

To show (*) we write out the difference quotients, first choosing \( h \) real and second choosing \( h \) purely imaginary. To get the Cauchy–Riemann equations one equates the real and imaginary parts of the two formulas for \( f'(z_0) \).

**Problem 6.** Suppose \( f : \mathbb{C} \to \mathbb{C} \) is holomorphic. Show the following:
- if \( \text{Re} (f) \) is constant, then \( f \) is constant,
- if \( \text{Im} (f) \) is constant, then \( f \) is constant,
- if \( |f| \) is constant, then \( f \) is constant.

Use the Cauchy–Riemann equations.

**Problem 7.** Let \( \{a_n\}_{n=1}^N \) and \( \{b_n\}_{n=1}^N \) be finite sequences in \( \mathbb{C} \), and define \( B_k = \sum_{n=1}^k b_n \), with the convention \( B_0 = 0 \). Prove the summation by parts formula:
\[
\sum_{n=M}^N a_n b_n = a_N B_N - a_M B_{M-1} - \sum_{n=M}^{N-1} (a_{n+1} - a_n) B_n.
\]

**Problem 8.** Show the following:
- the power series \( \sum_n n z^n \) does not converge for any \( z \in \partial \mathbb{D} \),
- the power series \( \sum_n \frac{1}{n} z^n \) converges for all \( z \in \partial \mathbb{D} \),
- the power series \( \sum_n \frac{1}{n} z^n \) converges for all \( z \in \partial \mathbb{D} \) except for \( z = 1 \).

The first series diverges by the “divergence test”. The second series converges absolutely by comparison with \( \sum_n \frac{1}{n} \). For the series, divergence at \( z = 1 \) is well known. For \( z \neq 1 \) we sum by parts and use the fact that for \( z \neq 1 \) we can write
\[
\left| \sum_{k=1}^n z^k \right| = \left| \frac{z - z^{n+1}}{1 - z} \right| \leq \frac{2}{|1 - z|}.
\]

This yields
\[
\left| \sum_{n=M}^N \frac{z^n}{n} \right| = \left| \frac{1}{N} \frac{z - z^{N+1}}{1 - z} - \frac{1}{M} \frac{z - z^M}{1 - z} - \sum_{n=M}^{N-1} \frac{-1}{n(n+1)} \frac{z - z^{n+1}}{1 - z} \right|
\leq \frac{2}{|1 - z|} \left( \frac{1}{N} + \frac{1}{M} + \sum_{n=M}^{N-1} \frac{1}{n^2} \right).
\]

Since \( \sum_n \frac{1}{n^2} \) converges, the quantity above tends to zero as \( M, N \to \infty \).