Math 185 - Spring 2015 - Homework 1 - Solution sketches

Hard copy due: Tuesday, February 3 at 11am.

Notation. $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $\partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$.

Problem 1. For all $z \in \mathbb{C}\backslash\{0\}$ there exists a unique $w \in \mathbb{C}\backslash\{0\}$ such that $zw = 1$, which we denote by $\frac{1}{z}$ or z^{-1}. Given $z = x + iy \in \mathbb{C}\backslash\{0\}$, compute the real and imaginary parts of z^{-1}.

If $z = x + iy$ then $z^{-1} = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2}$.

Problem 2. Describe the following sets in $\mathbb{C}$ geometrically and draw a picture of each.

- $\{z \in \mathbb{C} : |z - a| = |z - b|\}$, where $a, b \in \mathbb{C}$,
- $\{z \in \mathbb{C} : \mathrm{Re}\,(z) > 0\}$,
- $\{z \in C : \mathrm{Re}\,(az + b) > 0\}$, where $a, b \in \mathbb{C}$,
- $\{z \in \mathbb{C} : |z| = \mathrm{Re}\,(z) + 1\}$.

I'll describe the sets as subsets of the plane. The first set is the line of points equidistant to (a_1, a_2) and (b_1, b_2). The second set is the right-half plane. The third set is the set of points such that $a_2 y < a_1 x + b_1$ (also a half plane). The fourth set is the parabola $y^2 = 2x + 1$.

Problem 3. Define $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = \bar{z}$. Use the **definition of the derivative** to show that f is not holomorphic at any point.

Note $\frac{f(z+h)-f(z)}{h} = \frac{\bar{h}}{h}$. If we send $h \to 0$ along the real axis we get 1. If we send $h \to 0$ along the imaginary axis we get -1. Thus the limit as $h \to 0$ does not exist.

Problem 4. Fix $w \in \mathbb{D}$ and define the **Blaschke factor**

$$F(z) = \frac{w - z}{1 - \bar{w}z} \quad \text{for} \quad z \in \mathbb{D}.$$

Show the following:

- $F : \mathbb{D} \to \mathbb{D}$, and $F : \partial\mathbb{D} \to \partial\mathbb{D}$,
- F is a bijection on $\mathbb{D}$,
- F is holomorphic on $\mathbb{D}$.

Writing $z = re^{i\theta}$ we find

$$F(z) = e^{i\theta}\frac{w_0 - r}{1 - \bar{w}_0 r}, \qquad \text{where} \qquad w_0 = e^{-i\theta}w.$$

As $|e^{\pm i\theta}| = 1$, we see that without loss of generality we may assume $z \in \mathbb{R}$.

Now for $z \in \mathbb{R}$ the statement $|F(z)|^2 \leq 1$ can be seen to be equivalent to

$$|w|^2 + z^2 - 2z\mathrm{Re}\,(w) \leq 1 + z^2|w|^2 - 2z\mathrm{Re}\,(w),$$

or equivalently

$$|w|^2(1 - z^2) \leq 1 - z^2.$$

This inequality holds (strictly) when $|z| < 1$, while equality holds if $|z| = 1$.

To see that F is a bijection we note that it is its own inverse.

To see that F is holomorphic we note that it is a quotient of holomorphic functions and the denominator is never zero, since $|\bar{w}z| = |w|\,|z| < 1$.

Problem 5. Let $f : \mathbb{C} \to \mathbb{C}$ and define $u, v : \mathbb{R}^2 \to \mathbb{R}$ by

$$u(x, y) = \mathrm{Re}\,[f(x + iy)], \quad v(x, y) = \mathrm{Im}\,[f(x + iy)].$$

Suppose f is holomorphic at some $z_0 = x_0 + iy_0 \in \mathbb{C}$.

- Use the **definition of the derivative** to show that

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0) \quad \text{and} \quad f'(z_0) = -i\frac{\partial u}{\partial y}(x_0, y_0) + \frac{\partial v}{\partial y}(x_0, y_0). \quad (*)$$

- Use $(*)$ to derive the Cauchy–Riemann equations.

To show $(*)$ we write out the difference quotients, first choosing h real and second choosing h purely imaginary. To get the Cauchy–Riemann equations one equates the real and imaginary parts of the two formulas for $f'(z_0)$.

Problem 6. Suppose $f : \mathbb{C} \to \mathbb{C}$ is holomorphic. Show the following:

- if $\mathrm{Re}\,(f)$ is constant, then f is constant,
- if $\mathrm{Im}\,(f)$ is constant, then f is constant,
- if $|f|$ is constant, then f is constant.

Use the Cauchy–Riemann equations.

Problem 7. Let $\{a_n\}_{n=1}^N$ and $\{b_n\}_{n=1}^N$ be finite sequences in $\mathbb{C}$, and define $B_k = \sum_{n=1}^k b_n$, with the convention $B_0 = 0$. Prove the **summation by parts formula**:

$$\sum_{n=M}^{N} a_n b_n = a_N B_N - a_M B_{M-1} - \sum_{n=M}^{N-1} (a_{n+1} - a_n) B_n.$$

$$\begin{aligned}
\sum_{n=M}^{N} a_n b_n + \sum_{n=M}^{N-1} (a_{n+1} - a_n) B_n &= \sum_{n=M}^{N} a_n (B_n - B_{n-1}) + \sum_{n=M}^{N-1} a_{n+1} B_n - \sum_{n=M}^{N-1} a_n B_n \\
&= \sum_{n=M}^{N} a_n B_n - \sum_{n=M}^{N} a_n B_{n-1} + \sum_{n=M}^{N-1} a_{n+1} B_n - \sum_{n=M}^{N-1} a_n B_n \\
&= a_N B_N - a_M B_{M-1}.
\end{aligned}$$

Problem 8. Show the following:

- the power series $\sum_n n z^n$ does not converge for any $z \in \partial\mathbb{D}$,
- the power series $\sum_n \frac{1}{n^2} z^n$ converges for all $z \in \partial\mathbb{D}$,
- the power series $\sum_n \frac{1}{n} z^n$ converges for all $z \in \partial\mathbb{D}$ except for $z = 1$.

The first series diverges by the "divergence test". The second series converges absolutely by comparison with $\sum_n \frac{1}{n^2}$. For the series, divergence at $z = 1$ is well known. For $z \neq 1$ we sum by parts and use the fact that for $z \neq 1$ we can write

$$\left| \sum_{k=1}^{n} z^k \right| = \left| \frac{z - z^{n+1}}{1 - z} \right| \leq \frac{2}{|1 - z|}.$$

This yields

$$\begin{aligned}
\left| \sum_{n=M}^{N} \frac{z^n}{n} \right| &= \left| \frac{1}{N} \frac{z - z^{N+1}}{1 - z} - \frac{1}{M} \frac{z - z^M}{1 - z} - \sum_{n=M}^{N-1} \frac{-1}{n(n+1)} \frac{z - z^{n+1}}{1 - z} \right| \\
&\leq \frac{2}{|1 - z|} \left(\frac{1}{N} + \frac{1}{M} + \sum_{n=M}^{N-1} \frac{1}{n^2} \right).
\end{aligned}$$

Since $\sum_n \frac{1}{n^2}$ converges, the quantity above tends to zero as $M, N \to \infty$.