Abstract. These notes are meant to supplement the lectures for Math 126 (Introduction to PDE) in the Fall of 2014 at the University of California, Berkeley.

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1. Introduction/Background

1.1. Notation. The standard basis for $\mathbb{R}^d$ is denoted by $\{e^1, \ldots, e^d\}$, with $e^1 = (1, 0, \ldots, 0)$, etc. For $x \in \mathbb{R}^d$ we write $x = (x_1, \ldots, x_d)$ to denote $x = x_1e^1 + \cdots + x_de^d$. We call $x_1, \ldots, x_d$ the components of $x$.

The *inner product* (or *dot product*) of two vectors $x, y \in \mathbb{R}^d$ is given by $x \cdot y = x_1y_1 + \cdots + x_dy_d$. The norm of a vector $x \in \mathbb{R}^d$ is given by $|x| = \sqrt{x \cdot x}$.

For $x \in \mathbb{R}^d$ and $r > 0$, the *open ball centered at $x$ of radius $r$* is given by

$$B_r(x) = \{ y \in \mathbb{R}^d : |x - y| < r \}.$$

We use the notation $\alpha_d = \text{Volume}(B_1(0))$. Thus $\alpha_1 = 2$, $\alpha_2 = \pi$, $\alpha_3 = \frac{4}{3}\pi$, and so on. For general dimension $d$, there is a formula for $\alpha_d$ in terms of the Gamma function, but we will not need it. Note that we have $\text{Volume}(B_r(x)) = r^d \alpha_d$ for any $x \in \mathbb{R}^d$ and $r > 0$. We also have that $\text{Area}(\partial B_x(r)) = dr^{d-1} \alpha_d$.

A set $S \subset \mathbb{R}^d$ is *open* if for any $x \in \Omega$, there exists $r > 0$ such that $B_r(x) \subset S$. A set $S \subset \mathbb{R}^d$ is *closed* if its complement $\mathbb{R}^d \setminus S = \{ x \in \mathbb{R}^d : x \notin S \}$ is open. The *closure* of a set $S \subset \mathbb{R}^d$, denoted $\overline{S}$, is the intersection of all closed sets containing $S$. A set $S \subset \mathbb{R}^d$ is *compact* if it is closed and bounded.\(^1\) A set $\Omega \subset \mathbb{R}^d$ is *connected* if $\Omega$ cannot be written as a disjoint union of non-empty open sets.

We will use several notations for the partial derivatives of a function $u : \mathbb{R}^d \to \mathbb{R}$. For example,

$$\frac{\partial u}{\partial x_i} = \partial_i u = \partial_x u = u_{x_i}.$$

We denote the *gradient* of a function $u : \mathbb{R}^d \to \mathbb{R}$ by $\nabla u = (\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_d})$.

The *divergence* of a vector field $F : \mathbb{R}^d \to \mathbb{R}^d$ is denoted by $\nabla \cdot F$ or $\text{div} F$. The divergence is defined by $\text{div} F = \frac{\partial F_1}{\partial x_1} + \cdots + \frac{\partial F_d}{\partial x_d}$, where $F_1, \ldots, F_d$ denote the components of $F$.

The *Laplacian* of a function $u : \mathbb{R}^d \to \mathbb{R}$ is defined by $\Delta u = \text{div} \nabla u$. That is,

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_d^2}.$$

The Laplacian is an example of a differential operator. Other important examples include the *heat operator* $\partial_t - \Delta$ and the *wave operator* (or d’Alembertian) $\Box = \partial_t^2 - \Delta$.

A function is *smooth* if it has continuous derivatives of every order. We denote this by writing $f \in C^\infty$. The *support* of a function is defined by

$$\text{supp}(f) = \{ x : f(x) \neq 0 \}.$$

If $f \in C^\infty$ and $\text{supp}(f)$ is compact, we write $f \in C_0^\infty$.

For a set $\Omega \subset \mathbb{R}^d$ with smooth boundary $\partial \Omega$, we denote the outward-pointing unit normal vector at $x \in \partial \Omega$ by $n(x) = (n_1(x), \ldots, n_d(x))$.

1.2. Background results.

**Theorem 1.1** (Gauss–Green formula). For $i = 1, \ldots, d$,

$$\int_\Omega \frac{\partial u}{\partial x_i} \, dx = \int_{\partial \Omega} u n_i \, dS.$$

\(^1\)This is actually a theorem about compact sets in $\mathbb{R}^d$, not a definition of the word ‘compact’ (cf. the Heine–Borel Theorem).
Proposition 1.2 (Change of variables).

- For any \( x_0 \in \mathbb{R}^d \),
  \[
  \int_{\mathbb{R}^d} f(x + x_0) \, dx = \int_{\mathbb{R}^d} f(x) \, dx.
  \]
- For any \( c \in \mathbb{R} \setminus \{0\} \),
  \[
  \int_{\mathbb{R}^d} f(cx) \, dx = |c|^{-d} \int_{\mathbb{R}^d} f(x) \, dx.
  \]

Theorem 1.3 (Polar coordinates).

\[
\int_{\mathbb{R}^d} f(x) \, dx = \int_{0}^{\infty} \int_{\partial B_1(0)} f(r\theta) \, dS(\theta) \, r^{d-1} \, dr.
\]

1.3. Convolutions and distributions.

Definition 1.4. The convolution of \( f \) and \( g \) is defined by

\[
(f * g)(x) = \int_{\mathbb{R}^d} f(x-y)g(y) \, dy.
\]

We have \( f * g = g * f \) and \( \partial_i(f * g) = (\partial_i f) * g = f * (\partial_i g) \).

Theorem 1.5 (Approximate identities). Suppose \( \phi \in C_c^\infty(\mathbb{R}^d) \) satisfies \( \phi \geq 0 \), \( \int \phi \, dx = 1 \) and \( \text{supp}(\phi) \subset B_1(0) \). For each \( n \geq 1 \) define \( \phi_n(x) = n^d \phi(nx) \). Then for \( f \in C_c^\infty \) and \( x \in \mathbb{R}^d \) we have \( f * \phi_n(x) \to f(x) \) as \( n \to \infty \).

Sketch of proof. As \( \int \phi_n = 1 \) for each \( n \), we may write \( f(x) = \int \phi_n(y) f(x) \, dy \). We then have

\[
(f(x) - f * \phi_n(x)) = \int \phi_n(y)[f(x) - f(x-y)] \, dy = \int n^d\phi(ny)[f(x) - f(x-y)] \, dy.
\]

For each \( n \), the region of integration is restricted to \( |y| \leq \frac{1}{n} \). Therefore the continuity of \( f \) implies that as \( n \to \infty \), we have \( f(x-y) \to f(x) \) for each \( y \) in the region of integration. It follows that the integral converges to 0 as \( n \to \infty \).

Remark 1.6. While it is convenient to state and prove the theorem above under the assumptions \( f, \varphi \in C_c^\infty \), these are certainly not the most general assumptions under which the statement above is true.

Definition 1.7. A distribution is a function \( u : C_c^\infty \to \mathbb{R} \) that is

- linear, that is, \( u(\alpha f + \beta g) = \alpha u(f) + \beta u(g) \) for \( \alpha, \beta \in \mathbb{R} \) and \( f, g \in C_c^\infty \),
- continuous, that is, if \( f_k \to f \) then \( u(f_k) \to u(f) \).

The set of distributions is denoted \( \mathcal{D} \).

The Dirac delta distribution is defined by \( \delta_0(f) = f(0) \).

The set of \( C_c^\infty \)-functions ‘lives inside’ the set of distributions \( \mathcal{D} \), in the sense that we may define an injective mapping \( T : C_c^\infty \to \mathcal{D} \). In particular, for \( u \in C_c^\infty \), we can define \( Tu \in \mathcal{D} \) as follows:

\[
Tu(f) = \int_{\mathbb{R}^d} u(x)f(x) \, dx \quad \text{for} \quad f \in C_c^\infty.
\]

Proving the injectivity of this mapping is actually one of the problems from Homework 1. Because this mapping is one-to-one, we simply identify the function \( u \in C_c^\infty \) with the distribution \( Tu \in \mathcal{D} \), and thus refer to \( u \) itself as a distribution, with

\[
u(f) = \int_{\mathbb{R}^d} u(x)f(x) \, dx \quad \text{for} \quad f \in C_c^\infty.
\]

\textsuperscript{2}I haven’t actually told you what \( f_k \to f \) means for \( C_c^\infty \) functions. However, to define this is beyond the scope of these notes.
In fact, more general classes of functions than $C_c^\infty$ live inside $\mathcal{D}$ in the sense described above. For example, any locally integrable function\(^3\) lives inside of $\mathcal{D}$ in this sense.

**Definition 1.8** (Derivatives of distributions). If $u \in \mathcal{D}$, we define $\partial_i u \in \mathcal{D}$ by $\partial_i u(f) = -u(\partial_i f)$.

For $f \in C_c^\infty$ and $x \in \mathbb{R}^d$, we define $\tilde{f}(y) = f(-y)$ and $\tau_x f(y) = f(y-x)$.

**Definition 1.9** (Convolution with distributions). If $f \in C_c^\infty$ and $u \in \mathcal{D}$ we define $f \ast u \in C^\infty$ by $f \ast u(x) = u(\tau_x f)$.

One has $f \ast \delta_0 = f$ for $f \in C_c^\infty$.

**Definition 1.10.** Let $\mathcal{L}$ be a differential operator. A distribution $\Phi \in \mathcal{D}$ is a **fundamental solution** for $\mathcal{L}$ if $\mathcal{L}(\Phi) = \delta_0$.

If $\Phi$ is a fundamental solution for $\mathcal{L}$, then the function $u = f \ast \Phi$ solves $\mathcal{L}(u) = f$.

### 2. Laplace/Poisson Equation

The equation $-\Delta u = 0$ is **Laplace’s equation**.

The equation $-\Delta u = f$ is **Poisson’s equation**.

Solutions to $-\Delta u = 0$ are called **harmonic** functions.

#### 2.1. Fundamental solution.

The **fundamental solution** for $-\Delta$ is given by

$$
\Phi(x) = \begin{cases} 
-\frac{1}{2} |x| & d = 1 \\
-\frac{1}{2\pi} \log |x| & d = 2 \\
\frac{1}{d(d-2)\alpha_d} |x|^{-(d-2)} & d \geq 3.
\end{cases}
$$

For $d \geq 3$, the solution to Poisson’s equation $-\Delta u = f$ on $\mathbb{R}^d$ is given by

$$
u(x) = f \ast \Phi(x) = \frac{1}{d(d-2)\alpha_d} \int_{\mathbb{R}^d} |x-y|^{-(d-2)} f(y) \, dy.
$$

**Computation from class.** Fix the dimension $d \geq 3$. To see that $\Phi$ is the fundamental solution for $-\Delta$, it suffices to prove

$$
\int_{\mathbb{R}^d} |x|^{-(d-2)} \Delta f(x) \, dx = -d(d-2)\alpha_d f(0)
$$

for $f \in C_c^\infty$.

First let $\varepsilon > 0$ and write

$$
\int_{\mathbb{R}^d} |x|^{-(d-2)} \Delta f(x) \, dx = \int_{I} |x|^{-(d-2)} \Delta f(x) \, dx + \int_{\mathbb{R}^d \setminus B(0)} |x|^{-(d-2)} \Delta f(x) \, dx.
$$

For $I$, first use polar coordinates:

$$
\int_{B(0)} |x|^{-(d-2)} \, dx = \int_0^\varepsilon \int_{\partial B(0)} r^{d+2} \, dS \, dr = \alpha_d \int_0^\varepsilon r \, dr = \frac{\alpha_d \varepsilon^2}{2}.
$$

So

$$
|I| \leq \max_{x \in \mathbb{R}^d} |\Delta f(x)| \cdot \frac{\alpha_d \varepsilon^2}{2} \leq c \varepsilon^2.
$$

\(^3\)A function $f$ is locally integrable if $\int_K |f| \, dx < \infty$ for any compact set $K$. 
For II, we integrate by parts:

\[
\int_{\mathbb{R}^d \setminus B_\varepsilon(0)} |x|^{-(d-2)} \Delta f(x) \, dx = - \int_{\mathbb{R}^d \setminus B_\varepsilon(0)} \nabla (|x|^{-(d-2)}) \cdot \nabla f(x) \, dx \quad \text{III}
\]

\[
+ \int_{\partial B_\varepsilon(0)} |x|^{-(d-2)} \nabla f(x) \cdot n(x) \, dS. \quad \text{IV}
\]

We have

\[
|IV| \leq \varepsilon^{-(d-2)} \max_{x \in \mathbb{R}^d} |\nabla f(x)| \cdot \text{Area}(\partial B_\varepsilon(0)) \leq c \varepsilon^{-(d-2)} \varepsilon^{d-1} \leq c \varepsilon.
\]

For III we integrate by parts again:

\[
III = \int_{\mathbb{R}^d \setminus B_\varepsilon(0)} \Delta (|x|^{-(d-2)}) f(x) \, dx \quad (\equiv 0)
\]

\[
- \int_{\partial B_\varepsilon(0)} \nabla (|x|^{-(d-2)}) f(x) \cdot n(x) \, dS. \quad \text{V}
\]

For \( x \in \partial B_\varepsilon(0) \) we have

\[
\nabla (|x|^{-(d-2)}) = -(d-2) |x|^{-d} x = -(d-2) \varepsilon^{-d} x,
\]

\[
n(x) = -\frac{x}{|x|} = -\varepsilon^{-1} x,
\]

\[
\nabla (|x|^{-(d-2)}) \cdot n(x) = (d-2) \varepsilon^{-d-1} |x|^2 = (d-2) \varepsilon^{-(d-1)}.
\]

Recall \( \alpha_d = \text{Volume}(B_1(0)) \), so that \( \text{Area}(\partial B_\varepsilon(0)) = d \varepsilon^{d-1} \alpha_d \). Thus

\[
V = -(d-2) \varepsilon^{-(d-1)} \int_{\partial B_\varepsilon(0)} f(x) \, dS = -(d-2) \alpha_d \frac{1}{\text{Area}(\partial B_\varepsilon(0))} \int_{\partial B_\varepsilon(0)} f(x) \, dS.
\]

Putting it all together, we find that for any \( \varepsilon > 0 \) we have

\[
\int |x|^{-(d-2)} \Delta f(x) \, dx = -d(d-2) \alpha_d \frac{1}{\text{Area}(\partial B_\varepsilon(0))} \int_{\partial B_\varepsilon(0)} f(x) \, dS + O(\varepsilon + \varepsilon^2).
\]

We send \( \varepsilon \to 0 \) to find

\[
\int |x|^{-(d-2)} \Delta f(x) \, dx = -d(d-2) \alpha_d f(0),
\]

as was needed to show. \( \square \)

2.2. **Green’s function.** Let \( \Omega \subset \mathbb{R}^d \) be an open bounded set with smooth boundary \( \partial \Omega \). Suppose \( e : \overline{\Omega} \times \Omega \to \mathbb{R} \) is a function satisfying

\[
\begin{cases}
-\Delta_y e(x, y) = 0 & \text{for } x, y \in \Omega \\
e(x, y) = \Phi(x - y) & \text{for } x \in \Omega, \ y \in \partial \Omega.
\end{cases}
\]

Then the function

\[
G(x, y) = \Phi(x - y) - e(x, y)
\]

is called the **Green’s function** for \( \Omega \). It is a symmetric function of \( x \) and \( y \), that is, \( G(x, y) = G(y, x) \).

The solution to the Dirichlet boundary-value problem

\[
\begin{cases}
-\Delta u = f & \text{on } \Omega \\
u = g & \text{on } \partial \Omega
\end{cases}
\]

is given by

\[
u(x) = \int_{\Omega} G(x, y) f(y) \, dy - \int_{\partial \Omega} g(y) \nabla_y G(x, y) \cdot n(y) \, dS.
\]
Continuation of class notes. In class, we saw that

\[ u(x) = \int_{\Omega} G(x, y) f(y) \, dy \quad \text{solves} \quad \begin{cases} -\Delta u = f & \text{on } \Omega \\ u = 0 & \text{on } \partial \Omega \end{cases} \quad (\ast_1) \]

We can actually use the solution to \((\ast_1)\) to find the solution to

\[ \begin{cases} -\Delta u = 0 & \text{on } \Omega \\ u = g & \text{on } \partial \Omega. \end{cases} \quad (\ast_2) \]

Indeed, if \(v\) solves

\[ \begin{cases} -\Delta v = -\Delta g & \text{on } \Omega \\ v = 0 & \text{on } \partial \Omega, \end{cases} \]

then

\[ u(x) = g(x) - v(x) \quad \text{solves} \quad (\ast_2), \]

since we have \(-\Delta u = 0\) on \(\Omega\) and \(u = g\) on \(\partial \Omega\).

However, we can use our formula for solutions to \((\ast_1)\) and integrate by parts to find

\[ v(x) = -\int_{\Omega} G(x, y) \Delta g(y) \, dy \]

\[ = \int_{\Omega} \nabla_y G(x, y) \cdot \nabla g(y) \, dy - \int_{\partial \Omega} \mathbf{G}(x, y) \nabla g(y) \cdot \mathbf{n}(y) \, dS \]

\[ = -\int_{\Omega} \Delta_y G(x, y) g(y) \, dy + \int_{\partial \Omega} g(y) \nabla_y G(x, y) \cdot \mathbf{n}(y) \, dS \]

\[ = \int_{\Omega} \delta_0(x - y) g(y) \, dy + \int_{\partial \Omega} g(y) \nabla_y G(x, y) \cdot \mathbf{n}(y) \, dS \]

\[ = g(x) + \int_{\partial \Omega} g(y) \nabla_y G(x, y) \cdot \mathbf{n}(y) \, dS. \]

Thus we have the following formula for the solution to \((\ast_2)\):

\[ u(x) = g(x) - v(x) = -\int_{\partial \Omega} g(y) \nabla_y G(x, y) \cdot \mathbf{n}(y) \, dS. \]

\[ \square \]

2.3. Examples of Green’s functions.

Example 2.1 (Half-space). Let \(\Omega = \mathbb{R}^d_+ = \{ x \in \mathbb{R}^d : x_d \geq 0 \}\). Then \(\partial \Omega = \{ x \in \mathbb{R}^d : x_d = 0 \}\). For a point \(x \in \Omega\) define the reflection of \(x\) by

\[ \tilde{x} = (x_1, \ldots, x_{d-1}, -x_d). \]

The Green’s function for \(\Omega\) is given by

\[ G(x, y) = \Phi(x - y) - \Phi(\tilde{x} - y). \]

Example 2.2 (Unit ball). Let \(\Omega = B_1(0)\), so that \(\partial \Omega = \partial B_1(0)\). For a point \(x \in \Omega \setminus \{0\}\) define the inversion of \(x\) by

\[ \tilde{x} = \frac{x}{|x|^2}. \]

The Green’s function for \(\Omega\) is given by

\[ G(x, y) = \Phi(x - y) - \Phi(|x|(\tilde{x} - y)). \]
2.4. Mean value property. Let \( \Omega \subset \mathbb{R}^d \) be an open set and suppose \( u \) satisfies \( -\Delta u = 0 \) on \( \Omega \). Let \( x \in \Omega \) and suppose \( R > 0 \) is such that \( \overline{B_R(x)} \subset \Omega \). Then
\[
u(x) = \frac{1}{\text{Area}[\partial B_R(x)]} \int_{\partial B_R(x)} u(y) \, dS = \frac{1}{\text{Volume}[B_R(x)]} \int_{B_R(x)} u(y) \, dy.
\]
This is called the mean value property. Thus harmonic functions satisfy the mean value property. The converse is true as well.

2.5. Maximum principle. Suppose \( u \) is smooth and \( \Omega \subset \mathbb{R}^d \) is open, bounded, and connected. Suppose \( -\Delta u = 0 \) in \( \Omega \), and let \( M := \max_{x \in \Omega} u(x) \). If there exists \( x_0 \in \Omega \) such that \( u(x_0) = M \), then \( u \equiv M \) in \( \Omega \).

That is, non-constant harmonic functions attain their maximum value on the boundary.

2.6. Uniqueness. Let \( \Omega \subset \mathbb{R}^d \) be open, bounded, and connected. Suppose \( u_1 \) and \( u_2 \) are smooth solutions to
\[
\begin{align*}
& -\Delta u = f \quad \text{on} \ \Omega \\
& u = g \quad \text{on} \ \partial \Omega.
\end{align*}
\]
Then \( u_1 \equiv u_2 \) on \( \Omega \).

3. Heat equation

3.1. Fundamental solution. The fundamental solution for the heat equation is defined by
\[
\Phi(t, x) = \begin{cases}
(4\pi t)^{-d/2}e^{-|x|^2/4t} & (t, x) \in (0, \infty) \times \mathbb{R}^d \\
0 & (t, x) \in (-\infty, 0) \times \mathbb{R}^d.
\end{cases}
\]
The solution to the initial-value problem
\[
\begin{align*}
\partial_t u - \Delta u &= 0 \quad (t, x) \in (0, \infty) \times \mathbb{R}^d \\
u(0, x) &= f(x) \quad x \in \mathbb{R}^d
\end{align*}
\]
is given by
\[
u(t, x) = [f * \Phi(t, \cdot)](x) = (4\pi t)^{-d/2} \int_{\mathbb{R}^d} e^{-|x-y|^2/4t} f(y) \, dy.
\]

3.2. Mean value property. For \( \Omega \subset \mathbb{R}^d \) and \( T > 0 \) we define \( \Omega_T \subset \mathbb{R} \times \mathbb{R}^d \) by \( \Omega_T = (0, T] \times \Omega \). We let \( \Gamma_T = \overline{\Omega} \setminus \Omega_T \). For \( (t, x) \in \mathbb{R} \times \mathbb{R}^d \) and \( r > 0 \) we define
\[
E_r(t, x) = \{(s, y) \in \mathbb{R} \times \mathbb{R}^d \mid s \leq t \quad \text{and} \quad \Phi(t-s, x-y) \geq \frac{1}{r^d}\}.
\]
Suppose \( u \) is a smooth solution to \( u_t - \Delta u = 0 \) on \( \Omega_T \) and \( E_r(t, x) \subset \Omega_T \). Then
\[
u(t, x) = \frac{1}{4r^d} \int_{E_r(t, x)} u(s, y) \frac{|x-y|^2}{(t-s)^2} \, dy \, ds.
\]

3.3. Maximum principle. Suppose \( u \) is smooth and \( \Omega \subset \mathbb{R}^d \) is open, bounded, and connected. Suppose \( u_t - \Delta = 0 \) on \( \Omega_T \) for some \( T > 0 \). Let \( M := \max_{(t, x) \in \Omega_T} u(t, x) \). If there exists \( (t_0, x_0) \in \Omega_T \) such that \( u(t_0, x_0) = M \), then \( u(t, x) \equiv M \) on \( \overline{\Omega_{t_0}} \).

3.4. Uniqueness. Let \( \Omega \subset \mathbb{R}^d \) be open, bounded, and connected, and let \( T > 0 \). Suppose \( u_1 \) and \( u_2 \) are smooth solutions to
\[
\begin{align*}
& u_t - \Delta u = f \quad \text{on} \ \Omega_T \\
& u = g \quad \text{on} \ \Gamma_T.
\end{align*}
\]
Then \( u_1 \equiv u_2 \) on \( \Omega_T \).

4. Wave equation

The operator \( \Box = \frac{\partial^2}{\partial t^2} - \Delta \) is called the d'Alembertian.
4.1. **Solution in dimensions** $d = 1, 2, 3$. We consider the initial-value problem

\[
\begin{array}{ll}
\square u = 0 & (t, x) \in (0, \infty) \times \mathbb{R}^d \\
u(0, x) = f(x) & x \in \mathbb{R}^d \\
u_t(0, x) = g(x) & x \in \mathbb{R}^d
\end{array}
\]

The solution in dimension $d = 1$ is given by the **d’Alembert formula**

\[
u(t, x) = \frac{1}{2} [f(x + t) + f(x - t)] + \frac{1}{2} \int_{x-t}^{x+t} g(y) \, dy.
\]

The solution in dimension $d = 2$ is given by **Poisson’s formula**

\[
u(t, x) = \frac{1}{2\pi t^2} \int_{B_t(x)} \frac{tf(y) + t \nabla f(y) \cdot (y - x) + t^2g(y)}{(t^2 - |x - y|^2)^{1/2}} \, dy.
\]

The solution in dimension $d = 3$ is given by **Kirchhoff’s formula**

\[
u(t, x) = \frac{1}{4\pi t^3} \int_{\partial B_t(x)} f(y) + \nabla f(y) \cdot (y - x) + t g(y) \, dS(y).
\]

For $d = 1$ we can solve the PDE by introducing $\xi = x + t$ and $\eta = x - t$.

For $d = 3$ we reduced the problem to the $1d$ case by taking spherical means.

For $d = 2$ we used the method of descent.

4.2. **Fundamental solution.** The fundamental solution to the wave equation is the solution of

\[
\begin{array}{ll}
\square \Phi = 0, \\
\Phi(0) = 0, \quad \Phi_t(0) = \delta_0.
\end{array}
\]

The solution to the initial-value problem is then given by

\[
u(t, x) = f * \partial_t \Phi(t) + g * \Phi(t).
\]

In one dimension we found (by introducing $\xi = x + t$ and $\eta = x - t$ and looking for $\Phi$ of the form $\Phi(t, x) = v(\xi, \eta)$) that

\[
\Phi(t, x) = \frac{1}{2} [H(x + t) - H(x - t)]
\]

where $H$ is the Heaviside function,

\[
H(y) = \begin{cases} 
1 & y > 0 \\
0 & y < 0.
\end{cases}
\]

As $H' = \delta_0$, we find $\partial_t \Phi(t, x) = \frac{1}{2} [\delta_0(x + t) + \delta_0(x - t)]$, and so

\[
f * \partial_t \Phi(t) = \frac{1}{2} [f(x + t) + f(x - t)].
\]

On the other hand,

\[
g * \Phi(t) = \frac{1}{2} \int g(x - y) H(y + t) \, dy - \frac{1}{2} \int g(x - y) H(y - t) \, dy
\]

\[
= \frac{1}{2} \int_{-t}^{\infty} g(x - y) \, dy - \frac{1}{2} \int_{t}^{\infty} g(x - y) \, dy
\]

\[
= \frac{1}{2} \int_{-t}^{t} g(x - y) \, dy
\]

\[
= \frac{1}{2} \int_{x-t}^{x+t} g(y) \, dy.
\]
Thus we arrive at the d’Alembert formula

\[ u(t, x) = \frac{1}{2} [f(x + t) + f(x - t)] + \frac{1}{2} \int_{x-t}^{x+t} g(y) \, dy. \]

### 4.3. Huygens principle.

Suppose \( u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \) is a solution to \( \Box u = 0 \).

(Weak) If the dimension \( d \) is even, then \( u(t, x) \) depends on the initial data in \( B_t(x) \).

(Strong) If the dimension \( d \) is odd and \( d \geq 3 \), then \( u(t, x) \) depends on the initial data on \( \partial B_t(x) \).

### 4.4. Energy.

Let \( u \) be a smooth solution to \( \Box u = 0 \) such that \( u \) and its derivatives are integrable. We define the energy of \( u \) by

\[ E[u(t)] = \int_{\mathbb{R}^d} \frac{1}{2} [u_t(t, x)]^2 + \frac{1}{2} |\nabla u(t, x)|^2 \, dx. \]

Energy is conserved, that is, \( E[u(t)] \equiv 0 \).

### 4.5. Uniqueness.

Let \( \Omega \subset \mathbb{R}^d \) be an open, bounded set with smooth boundary \( \partial \Omega \). For \( T > 0 \), let \( \Omega_T = (0, T] \times \Omega \) and \( \Gamma_T = \overline{\Omega_T} \setminus \Omega_T \). Suppose \( u_1 \) and \( u_2 \) are solutions to

\[
\begin{cases}
\Box u = F & \text{on } \Omega_T \\
u = f & \text{on } \Gamma_T \\
u_t = g & \text{on } \{t = 0\} \times \Omega
\end{cases}
\]

Then \( u_1 \equiv u_2 \).

### 4.6. Finite speed of propagation.

Suppose \( \Box u = 0 \). Fix \( (t_0, x_0) \in (0, \infty) \times \mathbb{R}^d \) and define the cone

\[ C = \{(t, x) : 0 \leq t \leq t_0, \ |x - x_0| \leq t_0 - t\} \quad . \]

If \( u(0, x) = u_t(0, x) = 0 \) for \( x \in B_{t_0}(x_0) \), then \( u \equiv 0 \) for \( (t, x) \in C \).

## 5. Midterm 1 Review

**Background.**

- Calculus
  - chain rule
  - Gauss–Green formula, integration by parts
  - change of variables, polar coordinates
- Convolutions
  - definition, properties
  - approximate identities
- Distributions
  - definition, examples
  - derivatives of distributions

**Laplace/Poisson equation.**

\( \Delta u = f \)

- fundamental solution \( \Phi \) solves \( \Delta \Phi = \delta_0 \) on \( \mathbb{R}^d \)

\[ \Phi(x) = \begin{cases} 
-\frac{1}{2}|x| & d = 1 \\
-\frac{1}{2\pi} \log |x| & d = 2 \\
\frac{1}{d(d-2)\alpha_d} |x|^{-(d-2)} & d \geq 3. 
\end{cases} \]

solution to \( \Delta u = f \) on \( \mathbb{R}^d \) is given by \( u = f * \Phi \)
• Green’s function for $\Omega \subset \mathbb{R}^d$ is

$$G(x, y) = \Phi(x - y) - e(x, y),$$

where

$$\begin{cases} 
-\Delta_y e(x, y) = 0 & \text{for } x, y \in \Omega \\
e(x, y) = \Phi(x - y) & \text{for } x \in \Omega, \ y \in \partial \Omega.
\end{cases}$$

solution to

$$\begin{cases} 
-\Delta u = f & \text{on } \Omega \\
u = g & \text{on } \partial \Omega
\end{cases}$$

is given by

$$u(x) = \int_\Omega G(x, y) f(y) \, dy - \int_{\partial \Omega} g(y) \nabla_y G(x, y) \cdot n(y) \, dS.$$

• examples of Green’s functions: ball, half-space, interval, quarter-space, ...

• mean value property: if $-\Delta u = 0$ on $\Omega$ and $B_r(x) \subset \Omega$, then

$$u(x) = \frac{1}{\alpha_d R^d} \int_{B_r(x)} u \, dy = \frac{1}{d \alpha_d R^{d-1}} \int_{\partial B_r(x)} u \, dS.$$

• maximum principle:

suppose $-\Delta u = 0$ on $\Omega$ - open, bounded, connected

if $x_0 \in \Omega$ and $u(x_0) = \max_{\Omega T} u$, then $u$ is constant.

in particular $\max_{\Omega T} u = \max_{\partial \Omega} u$.

• uniqueness

**Heat equation.** $u_t - \Delta u = 0$

• fundamental solution $\Phi(t, x)$ solves

$$\begin{cases} 
\Phi_t - \Delta \Phi = 0 & (t, x) \in (0, \infty) \times \mathbb{R}^d \\
\Phi(0, \cdot) = \delta_0
\end{cases}$$

solution to

$$\begin{cases} 
u_t - \Delta u = 0 & (t, x) \in (0, \infty) \times \mathbb{R}^d \\
u(0, x) = f(x) & x \in \mathbb{R}^d
\end{cases}$$

is given by

$$u(t, x) = [f * \Phi(t, \cdot)](x) = (4\pi t)^{-d/2} e^{-\|x\|^2/4t}.$$

• infinite speed of propagation: if $f \geq 0$ then $u(t, x) > 0$ for all $(t, x) \in (0, \infty) \times \mathbb{R}^d$

• mean value property:

$$u(t, x) = \frac{1}{4\pi t^d} \int_{E_r(t, x)} u(s, y) \frac{|x - y|^2}{(t - s)^2} \, ds \, dy,$$

$E_r(t, x)$ is a ‘heat ball’

• maximum principle:

suppose $u_t - \Delta u = 0$ on $\Omega_T$ - with $\Omega$ open, bounded, connected

if $(t_0, x_0) \in \Omega_T$ and $u(t_0, x_0) = \max_{\Omega T} u$, then $u$ is constant on $\Omega_{t_0}$

• uniqueness

**Wave equation.** $\Box u = u_{tt} - \Delta u = 0$
• fundamental solution $\Phi(t,x)$ solves
  \[
  \begin{align*}
  \Box \Phi &= 0 \quad (t,x) \in (0,\infty) \\
  \Phi(0,\cdot) &= 0, \quad \Phi_t(0,\cdot) = \delta_0
  \end{align*}
  \]

• initial value problem
  \[
  \begin{align*}
  \Box u &= 0 \quad (t,x) \in (0,\infty) \\
  u(0) &= f, \quad u_t(0) = g
  \end{align*}
  \]

$d = 1$: d’Alembert formula:
  \[
  u(t,x) = \frac{1}{2} [f(x+t) + f(x-t)] + \frac{1}{2} \int_{x-t}^{x+t} g(y) \, dy
  \]

$d = 2$: Poisson’s formula:
  \[
  u(t,x) = \frac{1}{2\pi t^2} \int_{B_t(x)} \frac{tf(y) + t\nabla f(y) \cdot (y-x) + t^2g(y)}{(t^2 - |x-y|^2)^{1/2}} \, dy
  \]

$d = 3$: Kirchhoff’s formula:
  \[
  u(t,x) = \frac{1}{4\pi t^2} \int_{\partial B_t(x)} f(y) + \nabla f(y) \cdot (y-x) + t g(y) \, dS(y).
  \]

• Huygens principle:
  (strong version) for $d = 3$, $u(t,x)$ depends on initial data in $\partial B_t(x)$
  (weak version) for $d = 1, 2$, $u(t,x)$ depends on initial data in $B_t(x)$

• energy:
  \[
  \int \frac{1}{2} [u_t(t,x)]^2 + \frac{1}{2} |\nabla u(t,x)|^2 \, dx
  \]

• uniqueness

• finite speed of propagation: if $u(0,x) \equiv 0$ in $B_{t_0}(x_0)$ then $u \equiv 0$ in $C = \{(t,x) : 0 \leq t \leq t_0, \ |x-x_0| \leq t_0 - t\}$.

6. Fourier series and separation of variables

6.1. Boundary conditions. Consider an initial-value problem that takes place in a bounded domain $\Omega \subset \mathbb{R}^d$. In order for the problem to be well-posed one must impose some sort of boundary conditions. Two important examples are Dirichlet boundary conditions and Neumann boundary conditions. For Dirichlet boundary conditions one specifies the value of the solution $u(t,x)$ at each time $t$ and at $x \in \partial \Omega$. For Neumann boundary conditions one specifes the value of the ‘normal derivative’ of the solution $u$ at each time $t$ and at each $x \in \Omega$; that is, one specifies the value of $\nabla u(t,x) \cdot \mathbf{n}(x)$. We often consider the case of homogeneous boundary conditions, that is, when $u$ or $\nabla u \cdot \mathbf{n}$ is zero on the boundary.

6.2. Separation of variables. Consider an initial-value/boundary-value problem posed on $(t,x) \in (0,\infty) \times \Omega$. A separated solution is a solution of the form
  \[
  u(t,x) = p(t)q(x).
  \]

The idea of separation of variables is to try to write solutions to the problem as linear combinations of separated solutions.

For example, consider the heat equation $u_t = \Delta u$ with homogeneous boundary conditions (either Dirichlet or Neumann). Then $u(t,x) = p(t)q(x)$ is a separated solution provided
  \[
  \frac{-p'(t)}{p(t)} = \frac{-\Delta q(x)}{q(x)} = \lambda
  \]

for some constant $\lambda$.

This leads to the eigenvalue problem for $-\Delta$ with prescribed boundary conditions.
In particular, suppose $\lambda$ is an eigenvalue for $-\Delta$ with eigenfunction $q_\lambda$ (with $q_\lambda$ satisfying the boundary conditions). Then we get a separated solution

$$u_\lambda(t, x) = p_\lambda(t)q_\lambda(x) = e^{-\lambda t}q_\lambda(x).$$

More generally linear combinations such as

$$\sum_\lambda c_\lambda e^{-\lambda t}q_\lambda(x)$$

satisfy the PDE and boundary conditions. Then solving the initial-value/boundary-value problem comes down to being able to write the initial condition as a linear combination of eigenfunctions of the Laplacian.

6.3. Fourier series. Restricting to the case $\Omega = [0, L] \subset \mathbb{R}$, one finds that the eigenfunctions of $-\Delta$ are sines and cosines (depending on whether homogeneous Dirichlet or Neumann boundary conditions are imposed), with the eigenvalues given by $(\frac{n\pi}{L})^2$ (with $n > 0$ in the Dirichlet case, $n \geq 0$ in the Neumann case).

Thus in this special case the problem of separation of variables comes down to the question of expanding functions on $[0, L]$ (satisfying the boundary conditions) as linear combinations of sines or cosines.

These questions can be subsumed by the question of expanding periodic functions on $[-L, L]$ as linear combinations of complex exponentials. This is the topic of Fourier series.

The Fourier series of a periodic function $F : [-L, L]$ is defined by

$$F(x) = \sum_{n=-\infty}^{\infty} \langle F, e_n \rangle e_n(x),$$

where

$$e_n(x) = e^{\frac{in\pi x}{L}}, \quad \langle F, G \rangle := \frac{1}{2L} \int_{-L}^{L} F(x)\overline{G(x)} \, dx.$$ 

Under reasonable assumptions on $F$, one can show that the Fourier series converges to $F$.

The partial sums of the Fourier series are denoted

$$S_N f(x) = \sum_{n=-N}^{N} \langle f, e_n \rangle e_n(x).$$

We can write

$$S_N f(x) = f * D_N(x),$$

where $D_N$ is the Dirichlet kernel

$$D_N(x) = \frac{1}{2L} \sum_{n=-N}^{N} e_n(x), \quad x \in [-L, L].$$

The Cesàro means are defined by

$$\sigma_N f(x) = \frac{1}{N} \sum_{n=0}^{N-1} S_n f(x).$$

We can write

$$\sigma_N f(x) = f * F_N,$$

where $F_N$ is the Fejér kernel

$$F_N(x) = \frac{1}{N} \sum_{n=0}^{N-1} D_n(x).$$
One can use an approximate identity argument to show \( \sigma_N f(x) \to f(x) \). This shows that the Fourier series converge to \( f \) ‘on average’. It also shows that trigonometric polynomials (that is, linear combinations of the \( e_n \)) are dense in the space of continuous functions. This fact, plus the orthogonality of the \( e_n \), is a key ingredient in the proof of \( L^2 \)-convergence of Fourier series for continuous functions. That is,

\[
\lim_{N \to \infty} \| S_N f - f \| = 0,
\]

where

\[
\|f\| = \sqrt{\langle f, f \rangle}.
\]

As a consequence of \( L^2 \)-convergence, one has Parseval’s identity

\[
\sum_{n=-\infty}^{\infty} |\langle f, e_n \rangle|^2 = \|f\|^2
\]

and the Riemann–Lebesgue lemma

\[
\lim_{n \to \pm \infty} \langle f, e_n \rangle = 0.
\]

The Riemann–Lebesgue lemma and a more careful analysis of the Dirichlet kernel show that if \( f \) is differentiable at \( x_0 \) then \( S_N f(x_0) \to f(x_0) \) pointwise. However, if \( f \) is merely continuous then its Fourier series may diverge at a point.

### 7. Fourier transform

A multiindex \( \alpha = (\alpha_1, \ldots, \alpha_d) \) is an element of \( \mathbb{N}^d \), where \( \mathbb{N} = \{0, 1, 2, 3, \ldots\} \). Given a multi-index \( \alpha \) we define:

- \( |\alpha| = \alpha_1 + \cdots + \alpha_d \),
- \( x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d} \) for \( x \in \mathbb{R}^d \)
- \( \partial^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}} \)

We define Schwartz space by

\[
S = S(\mathbb{R}^d) = \{ f \in C^\infty(\mathbb{R}^d) : x^\alpha \partial^\beta f \text{ is bounded for all multiindices } \alpha, \beta \}.
\]

For \( f \in S(\mathbb{R}^d) \) we define \( \hat{f} : \mathbb{R}^d \to \mathbb{C} \) by

\[
\hat{f}(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{-ix \cdot \xi} \, dx.
\]

We call \( \hat{f} \) the Fourier transform of \( f \). We also write \( \hat{f} = \mathcal{F} f \).

The following identities hold:

- if \( g(x) = \partial^\alpha f(x) \) then \( \hat{g}(\xi) = (i\xi)^\alpha \hat{f}(\xi) \)
- if \( g(x) = (-ix)^\alpha f(x) \) then \( \hat{g}(\xi) = \partial^\alpha \hat{f}(\xi) \).

Using these identities, one can show that \( \mathcal{F} : S \to S \). In fact, one can show that \( \mathcal{F} \) is a bijection on Schwartz space and that the Fourier inversion formula holds:

\[
f(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \hat{f}(\xi) e^{ix \cdot \xi} \, d\xi \quad \text{for } f \in S.
\]

The inverse Fourier transform \( \mathcal{F}^{-1} : S \to S \) is defined by

\[
\mathcal{F}^{-1} f(x) = \hat{f}(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(\xi) e^{ix \cdot \xi} \, d\xi.
\]
Furthermore the Fourier transform is a unitary transformation with respect to the $L^2$ inner product. That is, if we define

$$\langle f, g \rangle = \int_{\mathbb{R}^d} f(x) \overline{g(x)} \, dx,$$

then we have Plancherel’s theorem: $\langle \hat{f}, \hat{g} \rangle = \langle f, g \rangle$.

We have the multiplication identity

$$\int_{\mathbb{R}^d} \hat{f}(x) g(x) \, dx = \int_{\mathbb{R}^d} f(x) \overline{g(x)} \, dx$$

and the convolution identities

$$F(f * g) = (2\pi)^{d/2} \hat{f} \hat{g}$$

$$\iff$$

$$F^{-1}(\hat{f} \hat{g}) = (2\pi)^{-d/2} f * g.$$

A Fourier multiplier operator is an operator $T$ such that

$$\hat{Tf}(\xi) = m(\xi) \hat{f}(\xi)$$

for some function $m$. (We call $m$ the multiplier of $T$.) If $T$ is a Fourier multiplier operator, we may also write

$$Tf(x) = f * K(x), \quad K(x) = (2\pi)^{-d/2} \hat{m}(x).$$

7.1. Tempered distributions. A tempered distribution is a continuous linear functional $u : S(\mathbb{R}^d) \rightarrow \mathbb{C}$. The space of tempered distributions is denoted $S'$. For $u \in S'$ we define the Fourier transform of $u$, denoted $\hat{u} \in S'$, by

$$\hat{u}(f) = u(\hat{f}) \quad \text{for} \quad f \in S'.$$

The Fourier transform is a bijection on $S'$.

8. Duhamel’s principle

Consider a linear inhomogeneous PDE of the form

$$\begin{cases}
  u_t + Lu = F & (t, x) \in (0, \infty) \times \mathbb{R}^d \\
  u(0, f) = f(x) & x \in \mathbb{R}^d.
\end{cases}$$

Suppose that solutions to the homogeneous problem

$$\begin{cases}
  u_t + Lu = 0 & (t, x) \in (0, \infty) \times \mathbb{R}^d \\
  u(0, x) = f(x) & x \in \mathbb{R}^d
\end{cases}$$

are given by $u(t) = S(t)f$ for some operator $S(t)$ such that $S(t)^{-1} = S(-t)$ and $S(t+s) = S(t)S(s)$. Using ‘variation of parameters’ (i.e. looking for a solution of the form $u(t) = S(t)v(t)$) we can write the Duhamel formula for the solution to (1):

$$u(t) = S(t)f + \int_0^t S(t-s)F(s) \, ds.$$

9. Method of characteristics

Let $\Omega \subset \mathbb{R}^d$. Consider a first-order PDE

$$F(x, u, \nabla u) = 0 \quad \text{on} \quad \Omega$$

with boundary condition

$$u = g \quad \text{on} \quad \Gamma \subset \Omega.$$

The idea of the method of characteristics is to find a curve $x(s) \in \Omega$ emanating from some $x_0 \in \Gamma$ such that we can find the values of the solution $z(s) = u(x(s))$ along the curve $x(s)$. The curve $x(s)$ is called a characteristic.
If $F$ is linear, that is,

$$F(u, \nabla u) = a(x)u(x) + b(x) \cdot \nabla u(x),$$

then we derive the characteristic ODEs

$$\begin{align*}
\dot{x}(s) &= b(x(s)) \\
x(0) &= x_0 \in \Gamma \\
\dot{z}(s) &= -a(x(s))z(s), \\
z(0) &= g(x_0).
\end{align*}$$

If $F$ is quasilinear, that is,

$$F(x, u, \nabla u) = a(x, u(x)) + b(x, u(x)) \cdot \nabla u(x),$$

then we derive the characteristic ODEs

$$\begin{align*}
\dot{x}(s) &= b(x(s), z(s)) \\
x(0) &= x_0 \in \Gamma \\
\dot{z}(s) &= -a(x(s), z(s)) \\
z(0) &= g(x_0).
\end{align*}$$

If $F$ is fully nonlinear then one also has to derive an ODE for $p(s) = \nabla u(x(s))$. We did not pursue this case in class.

To employ the method of characteristics, one needs to solve the system of ODEs for $x(s)$ and $z(s)$. Given $x \in \Omega$, one needs to find $x_0 \in \Gamma$ and $s$ such that $x(s) = x$. Then $u(x) = z(s)$.

### 10. Scalar conservation laws

A scalar conservation law is a PDE of the form

$$\begin{align*}
\frac{\partial u}{\partial t} + \left[F(u)\right]_y &= 0 \quad (t, y) \in (0, \infty) \times \mathbb{R} \\
u(0, y) &= f(y) \quad y \in \mathbb{R}.
\end{align*}$$

We usually consider nice smooth $F$ that satisfies $F'' > \varepsilon > 0$.

For example, with $F(u) = \frac{1}{2}u^2$ we have Burger’s equation.

The characteristic ODEs for (2) are

$$\begin{align*}
\dot{x}(s) &= (1, F'(z(s))) \\
x(0) &= (0, y_0) \\
\dot{z}(s) &= 0 \\
z(0) &= g(y_0)
\end{align*}$$

Two potential issues arise: the crossing of characteristics (in which case one define shocks and cope with discontinuous solutions) and incomplete characteristics (in which case there is a non-uniqueness issue).

A bounded function $u : (0, \infty) \times \mathbb{R} \to \mathbb{R}$ is an integral solution to (2) if

$$\int_0^\infty \int_\mathbb{R} [u\varphi_t + F(u)\varphi_y] \, dy \, dt + \int_\mathbb{R} g(y)\varphi(0, y) \, dy = 0$$

for all $\varphi \in C_c^\infty((0, \infty) \times \mathbb{R})$.

The Rankine–Hugoniot condition for shocks is given by

$$F(u_\ell) - F(u_r) = s(u_\ell - u_r) \quad \text{on} \quad C = \{(t, y) : y = s(t)\},$$

where $C$ is a shock and $u_\ell, u_r$ are the limits of $u$ on $C$ from the left and right. The entropy condition is

$$F'(u_\ell) > s > F'(u_r).$$
11. CALCULUS OF VARIATIONS

Some PDE arise naturally when considering certain optimization problems. In particular solving a PDE may be equivalent to minimizing an appropriate functional.

Let

\[ \mathcal{A} = \{ v \in C^2(\overline{\Omega}) : v = g \text{ on } \partial\Omega \} \]

and define the Dirichlet energy functional

\[ E[v] = \frac{1}{2} \int_{\Omega} |\nabla v|^2 \, dx \quad \text{for} \quad v \in \mathcal{A}. \]

**Dirichlet’s principle** states that \( u \in \mathcal{A} \) satisfies

\[ E[u] = \min_{v \in \mathcal{A}} E[v] \]

if and only if \( u \) solves

\[ \begin{cases} -\Delta u = 0 & \text{on } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases} \]

That is, Laplace’s equation is the **Euler–Lagrange equation** associated to the Dirichlet energy functional.

12. MIDTERM 2 REVIEW

- **Boundary conditions**
  - Dirichlet
  - Neumann
- **Separation of variables**
  - eigenvalue problem for \(-\Delta\)
  - 1d heat/wave equations on the interval
- **Fourier series**
  - Dirichlet kernel
  - Fejer kernel
  - Cesáro means
  - \(L^2\) norm and inner product
  - orthogonality
  - Parseval’s identity
  - Riemann–Lebesgue lemma
  - application to PDE
- **Fourier transform**
  - Schwartz space
  - multiplication identity
  - convolution identity
  - Plancherel theorem
  - tempered distributions
  - application to PDE
  - Fourier multiplier and convolution operators
- **Duhamel’s principle**
- **Method of characteristics**
  - characteristic ODE
  - linear, quasilinear cases
- **Scalar conservation laws**
  - Burger’s equation
  - crossing characteristics, shocks
incomplete characteristics, non-uniqueness
– Rankine–Hugoniot condition
– entropy condition
– integral solution, entropy solution

• Calculus of variations
  – Euler–Lagrange equation
  – Dirichlet’s principle
  – constrained minimization
  – eigenvalue problem for $-\Delta$

13. Numerical methods

Often one wants to approximate the solution to a PDE by the solution to a discretized problem. One can (approximately) solve the discretized problem by using a computer (e.g. matlab, python).

In particular we consider grid points $(t_n, x_j)$ given by $t_n = n\Delta t$ and $x_j = j\Delta x$ for some mesh sizes $\Delta t$ and $\Delta x$. We approximate the solution to the continuous problem by the solution $u(t_n, x_j)$ to the discretized problem on the grid.

Motivated by the Taylor series expansion we can approximate derivatives using finite differences.

For example:

\[
\partial_x u(x_j) \approx \delta^-_x u(x_j) := \frac{u(x_j) - u(x_{j-1})}{\Delta x} \quad \text{(backward difference)}
\]
\[
\approx \delta^+_x u(x_j) := \frac{u(x_{j+1}) - u(x_j)}{\Delta x} \quad \text{(forward difference)}
\]
\[
\approx \delta_x u(x_j) := \frac{u(x_{j+1}) - u(x_{j-1})}{2\Delta x} \quad \text{(centered difference)}.
\]

The first two have errors like $\Delta x$, while the third has an error like $(\Delta x)^2$.

Similarly,

\[
\partial_{xx} u(x_j) \approx \delta^2_x u(x_j) := \frac{u(x_{j+1}) - 2u(x_j) + u(x_{j-1})}{(\Delta x)^2} \quad \text{(centered second difference)}
\]

with errors like $(\Delta x)^2$.

Of course one can approximate derivatives with respect to $t$ the same way.

One then defines a scheme for a given (let’s say linear) PDE, which (in principle) allows for the computation of $u(t_n, x_j)$. In practice, however, one uses a computer to compute, and so one can not truly compute $u(t_n, x_j)$. Instead, one can compute $\tilde{u}(t_n, x_j) = u(t_n, x_j) + e(t_n, x_j)$, where $e(t_n, x_j)$ records numerical error.

Whether or not one can expect the error $e$ to remain bounded is the question of numerical stability for the scheme.

One way to analyze this is via von Neumann stability analysis. The idea is to expand $e$ in a Fourier series. Because $u$ and $\tilde{u}$ satisfy the scheme and the problem is linear, then $e$ also satisfies the scheme. Hence one can use the recurrence relation of the scheme to study the behavior of the Fourier coefficients.

14. Quantum mechanics

The Schrödinger equation is

\[
\begin{cases}
  i\hbar \psi_t = -\frac{\hbar^2}{2m} \Delta \psi + V(x)\psi & (t, x) \in \mathbb{R} \times \mathbb{R}^d \\
  \psi(0, x) = \psi_0(x) & x \in \mathbb{R}^d,
\end{cases}
\]

where $i = \sqrt{-1}$ and $\hbar$ is Planck’s constant.
A particle of mass $m > 0$ in the presence of a potential $V(x)$ is modelled by an element $\Psi \in L^2$ (called the state of the particle), with the evolution of $\Psi$ governed by the Schrödinger equation.

The function $\Psi$ gives the position distribution of the particle, while $\Phi(\xi) := h^{-d/2}\hat{\Psi}(\xi/h)$ gives the momentum distribution. More precisely, the interpretation is as follows:

\[
\int_{\Omega} |\Psi(x)|^2 \, dx = \text{probability that the position is in } \Omega,
\]

\[
\int_{\Omega'} |\Phi(\xi)|^2 \, d\xi = \text{probability that the momentum is in } \Omega'.
\]

Observables are modelled by hermitian operators $Q$. Hermitian means $\langle \Psi, Q \Psi \rangle = \langle Q \Psi, \Psi \rangle$, where

\[
\langle f, g \rangle = \int_{\mathbb{R}^d} f(x)g(x) \, dx.
\]

In particular, the expectation value of $Q$ for a particle in state $\Psi$ is given by

\[
E(Q) = \langle \Psi, Q \Psi \rangle.
\]

The variance of $Q$ is computed by

\[
\sigma_Q^2 = \| (Q - E(Q)) \Psi \|^2,
\]

where $\| f \| = \sqrt{\langle f, f \rangle}$.

Position is modelled by the operator

\[
[x \Psi](x) = x \Psi(x),
\]

while momentum is modelled by the operator

\[
[p \Psi](x) = -i\hbar \nabla \Psi(x).
\]

Energy (also known as the Hamiltonian) is modelled by

\[
H \Psi = -\frac{\hbar^2}{2m} \Delta \Psi + V(x) \Psi.
\]

**Example.** (free particle) Suppose $V \equiv 0$. Then the Schrödinger equation becomes

\[
\begin{cases}
i\Psi_t = -k \Delta \Psi, & k = \frac{\hbar}{2m} \\
\Psi(0, x) = \Psi_0(x).
\end{cases}
\]

Applying the Fourier transform:

\[
i\hat{\Psi}_t(t, \xi) = k|\xi|^2 \hat{\Psi}(t, \xi).
\]

We find

\[
\hat{\Psi}(t, \xi) = e^{-ikt|\xi|^2} \hat{\Psi}(0, \xi) = e^{-ikt|\xi|^2} \hat{\Psi}_0(\xi).
\]

We therefore write

\[
\Psi(t) = e^{ikt\Delta} \Psi_0, \quad e^{ikt\Delta} = \mathcal{F}^{-1}e^{-ikt|\xi|^2} \mathcal{F}.
\]

We can compare this to the classical setting. In classical mechanics the trajectory of a free particle with initial position $x(0) = x_0$ and initial momentum $p(0) = p_0$ is given by $x(t) = x_0 + \frac{p_0}{m}t$ and $p(t) = p_0$.

In the quantum setting we cannot specify both $x_0$ and $p_0$ to arbitrary precision (this is a consequence of the uncertainty principle).

As a substitute, consider an initial state

\[
\Psi_0(x) = ce^{-|x-x_0|^2/2}e^{ix\cdot\xi_0}.
\]

Then

\[
\hat{\Psi}_0(\xi) = ce^{-|\xi-\xi_0|^2/2}e^{-i\xi\cdot x_0} \implies \Phi_0(\xi) = h^{-d/2}\hat{\psi}_0(\xi) = ch^{-d/2}e^{-|\xi-\xi_0|^2/2\hbar^2}e^{-i\xi\cdot x_0/h}.
\]

Thus $\Phi_0$ is concentrated where $\xi \sim p_0$ provided we choose $\xi_0 = \frac{p_0}{\hbar}$. 

From above we have \( \hat{\Psi}(t, \xi) = e^{-ikt|\xi|^2} \hat{\Psi}_0(\xi) \), so \( \hat{\Psi} \) remains concentrated where \( \xi \sim \xi_0 \).

Equivalently \( \Phi \) remains concentrated near \( \xi \sim p_0 \).

Taking the inverse Fourier transform we have

\[
\Psi(t, x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ix \cdot \xi} e^{-ikt|\xi|^2} \hat{\Psi}_0(\xi) \, d\xi
\]

Since \( \hat{\Psi}_0 \) is concentrated where \( \xi \sim \xi_0 \) we use Taylor’s theorem and approximate

\[
|\xi|^2 \approx |\xi_0|^2 + 2\xi_0 \cdot (\xi - \xi_0).
\]

Thus

\[
\Psi(t, x) \approx (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ix \cdot \xi} e^{-ikt|\xi_0|^2 + 2\xi_0 \cdot (\xi - \xi_0)} \hat{\Psi}_0(\xi) \, d\xi
\]

\[
= (2\pi)^{-d/2} e^{-ikt|\xi_0|^2} e^{ix \cdot \xi_0} \int_{\mathbb{R}^d} e^{i(x - 2tk\xi_0) \cdot (\xi - \xi_0)} \hat{\Psi}_0(\xi - \xi_0 + \xi_0) \, d\xi
\]

\[
= (2\pi)^{-d/2} e^{-ikt|\xi_0|^2 + ix \cdot \xi_0} \int_{\mathbb{R}^d} e^{i(x - 2tk\xi_0) \cdot (\xi - \xi_0)} \mathcal{F}[e^{-i\xi_0 \cdot x} \Psi_0](\xi - \xi_0) \, d\xi
\]

\[
= e^{-ikt|\xi_0|^2 + ix \cdot \xi_0} [e^{-i\xi_0 \cdot (x - 2tk\xi_0)} \Psi_0(x - 2tk\xi_0)]
\]

\[
= e^{ikt|\xi_0|^2} \Psi_0(x - 2tk\xi_0).
\]

Thus \( \Psi \) travels with speed \( 2k\xi_0 = \frac{\hbar \xi_0}{m} = \frac{p_0}{m} \), in agreement with the classical picture.
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