Problem 1. Let $\Omega \subset \mathbb{R}^d$ be an open, bounded set with smooth boundary $\partial \Omega$. Suppose $u$ is a smooth solution to
\[
\begin{cases}
  u_t - \Delta u = 0 & (t, x) \in (0, \infty) \times \Omega, \\
  u(0, x) = f(x) & x \in \Omega, \\
  \nabla u(t, x) \cdot n(x) = 0 & x \in \partial \Omega.
\end{cases}
\]
Show that
\[
\int_\Omega u(t, x) \, dx = \int_\Omega f(x) \, dx.
\]
\[
\frac{d}{dt} \int_\Omega u(t, x) \, dx = \int_\Omega \Delta u(t, x) \, dx = \int_{\partial \Omega} \nabla u(t, x) \cdot n(x) \, dS = 0.
\]

Problem 2. Let $\Omega \subset \mathbb{R}^d$ be an open, bounded set with smooth boundary. Consider the eigenvalue problem
\[-\Delta q(x) = \lambda q(x), \quad q : \overline{\Omega} \to \mathbb{C}\]
with either Dirichlet or Neumann boundary conditions, that is,
\[q(x) = 0 \text{ for } x \in \partial \Omega \quad \text{or} \quad \nabla q(x) \cdot n(x) = 0 \text{ for } x \in \partial \Omega.
\]
(i) Show that all eigenvalues are real.
(ii) Show that for any eigenvalue we can find a real-valued eigenfunction.
(iii) Suppose that $q_1$ is an eigenfunction with eigenvalue $\lambda_1$ and $q_2$ is an eigenfunction with eigenvalue $\lambda_2$. Show that if $\lambda_1 \neq \lambda_2$ then $q_1$ and $q_2$ are orthogonal, that is,
\[
\int_\Omega q_1(x)\overline{q_2(x)} \, dx = 0.
\]
If $f, g$ satisfy either of the boundary conditions then
\[
\int_\Omega \Delta f \overline{g} \, dx = -\int_\Omega \nabla f \cdot \nabla \overline{g} \, dx + \int_{\partial \Omega} \overline{\nabla f} \cdot n \, dS = \int_\Omega f \overline{\Delta g} \, dx - \int_{\partial \Omega} f \overline{\nabla g} \cdot n \, dS = \int_\Omega f \Delta \overline{g} \, dx.
\]
Suppose $q$ is an eigenvector with eigenvalue $\lambda$ satisfying either boundary conditions. Then
\[
\lambda \int_\Omega |q|^2 = \lambda \int_\Omega q\overline{q} = \int_\Omega (-\Delta q)\overline{q} = \int_\Omega q(-\Delta \overline{q}) = \overline{\lambda} \int_\Omega q\overline{q} = \overline{\lambda} \int_\Omega |q|^2
\]
As $q \neq 0$ we have $\int |q|^2 \neq 0$ so this implies $\lambda = \overline{\lambda}$, i.e. $\lambda \in \mathbb{R}$.

Now if $\lambda \in \mathbb{R}$ is an eigenvalue with corresponding eigenfunction $q$, then $-\Delta q = \lambda q$ gives $-\Delta \overline{q} = \overline{\lambda} \overline{q} = \lambda q$ so $\overline{q}$ is an eigenfunction corresponding to $\lambda$ as well. Thus $q + \overline{q} = 2\text{Re}(q)$ is a real-valued eigenfunction corresponding to $\lambda$.

If $q_1, q_2$ are eigenfunctions corresponding to $\lambda_1, \lambda_2$ then computing as above gives
\[
\lambda_1 \int_\Omega q_1 \overline{q_2} \, dx = \lambda_2 \int_\Omega q_1 \overline{q_2} \, dx = \lambda_2 \int_\Omega q_1 \overline{q_2} \, dx.
\]
As $\lambda_1 \neq \lambda_2$ this implies
\[
\int_\Omega q_1 \overline{q_2} \, dx = 0.
\]

Problem 3. Consider the eigenvalue problem with homogeneous Dirichlet boundary conditions:
\[
\begin{cases}
  -q''(x) = \lambda q(x), & x \in (0, L) \\
  q(0) = q(L) = 0.
\end{cases}
\]
(i) Show that 0 is not an eigenvalue.
(ii) Show that there are no negative eigenvalues.

(iii) Find the eigenvalues and associated eigenfunctions.

Suppose \( \lambda = 0 \). The solution to \( q'' = 0 \) is \( q(x) = ax + b \). Imposing the boundary conditions gives \( q(x) \equiv 0 \). Thus there can be no eigenfunction if \( \lambda = 0 \).

Suppose now \( \lambda < 0 \), say \( \lambda = -\mu^2 \). The general solution to \( q'' = \mu^2 q \) is given by \( q(x) = Ae^{\mu x} + Be^{-\mu x} \). Imposing the boundary conditions gives \( q(x) \equiv 0 \). Thus there can be no eigenfunction if \( \lambda < 0 \).

According to the previous problem, the eigenvalues are real, so the only other option is \( \lambda > 0 \).

For \( \lambda > 0 \), say \( \lambda = \mu^2 \), the general solution to \( q'' = -\mu^2 q \) is \( q(x) = A \cos(\mu x) + B \sin(\mu x) \). Imposing \( q(0) = 0 \) implies \( A = 0 \). Imposing \( q(L) = 0 \) implies \( \mu = \frac{n\pi}{L} \) for some \( n > 0 \).

Thus the eigenvalues are \( \left( \frac{n\pi}{L} \right)^2 \) and the corresponding eigenfunctions are \( q_n(x) = \sin\left( \frac{n\pi}{L} x \right) \) (for \( n > 0 \)).

**Problem 4.** Consider the eigenvalue problem with homogeneous Neumann boundary conditions:

\[
\begin{cases}
-q''(x) = \lambda q(x), & x \in (0, L) \\
q(0) = q'(L) = 0.
\end{cases}
\]

(i) Show that there are no negative eigenvalues.

(ii) Find the eigenvalues and associated eigenfunctions.

Suppose \( \lambda < 0 \) is an eigenvalue, say \( \lambda = -\mu^2 \). The general solution is again \( q(x) = Ae^{\mu x} + Be^{-\mu x} \). Imposing the boundary conditions gives \( q(x) \equiv 0 \). Thus there can be no eigenfunction if \( \lambda < 0 \).

According to the Problem 2, the eigenvalues are real, so the only other option is \( \lambda > 0 \).

For \( \lambda > 0 \), we have the constant eigenfunction \( q(x) \equiv 1 \).

According to the Problem 2, the eigenvalues are real, so the only other option is \( \lambda > 0 \).

For \( \lambda > 0 \), say \( \lambda = \mu^2 \), the general solution is \( q(x) = A \cos(\mu x) + B \sin(\mu x) \). Imposing \( q'(0) = 0 \) gives \( B = 0 \). Imposing \( q'(L) = 0 \) implies \( \mu = \frac{n\pi}{L} \) for some \( n > 0 \).

Thus the eigenvalues are \( \left( \frac{n\pi}{L} \right)^2 \) and the corresponding eigenfunctions are \( q_n(x) = \cos\left( \frac{n\pi}{L} x \right) \) (for \( n \geq 0 \)).

**Problem 5.** Suppose that for any smooth \( F : [-L, L] \to \mathbb{C} \) satisfying \( F(L) = F(-L) \) we can write

\[
F(x) = \sum_{n=-\infty}^{\infty} c_n e^{i \frac{n\pi}{L} x} \quad \text{for some} \quad c_n \in \mathbb{C}.
\]

(i) Show that if \( f : (0, L) \to \mathbb{R} \) is smooth and satisfies \( f(0) = f(L) = 0 \) then we can write

\[
f(x) = \sum_{n=1}^{\infty} a_n \sin\left( \frac{n\pi}{L} x \right) \quad \text{for some} \quad a_n \in \mathbb{R}.
\]

(ii) Show that if \( f : (0, L) \to \mathbb{R} \) is smooth and satisfies \( f'(0) = f'(L) = 0 \) then we can write

\[
f(x) = \sum_{n=0}^{\infty} b_n \cos\left( \frac{n\pi}{L} x \right) \quad \text{for some} \quad b_n \in \mathbb{R}.
\]

Let \( f : [0, L] \to \mathbb{R} \) satisfy \( f(0) = f(L) = 0 \). Define \( F : [-L, L] \to \mathbb{R} \) by \( F(x) = f(x) \) for \( x \geq 0 \) and \( F(x) = -f(-x) \) for \( x < 0 \). We write

\[
F(x) = \sum_{n=-\infty}^{\infty} c_n e^{i \frac{n\pi}{L} x} = c_0 + \sum_{n=1}^{\infty} [c_n + c_{-n}] \cos\left( \frac{n\pi}{L} x \right) + i[c_n - c_{-n}] \sin\left( \frac{n\pi}{L} x \right). \quad (*)
\]

Using \( F(x) = -F(-x) \) we can deduce \( c_0 = 0 \) and \( c_n = -c_{-n} \). Thus

\[
F(x) = \sum_{n=1}^{\infty} 2ic_n \sin\left( \frac{n\pi}{L} x \right).
\]
It remains to see that $2ic_n \in \mathbb{R}$. To see this we can use $F(x) = \overline{F(x)}$ to deduce that $2ic_n = \overline{2ic_n}$.

Thus $2ic_n = \overline{2ic_n}$, so that $2ic_n \in \mathbb{R}$.

Now let $f : [0, L] \to \mathbb{R}$ satisfy $f'(0) = f'(L) = 0$. Define $F : [-L, L] \to \mathbb{R}$ by $F(x) = f(x)$ for $x \geq 0$ and $F(x) = f(-x)$ for $x < 0$. Expanding $F$ as in (*), and using $F(x) = \overline{F(-x)}$ we can deduce $c_n = c_{-n}$ so that

$$F(x) = c_0 + \sum_{n=1}^{\infty} 2c_n \cos\left(\frac{n\pi}{L} x\right).$$

It remains to see that $c_n \in \mathbb{R}$ for $n \geq 0$. Again we can use $F(x) = \overline{F(x)}$ to deduce that $2c_n = \overline{2c_n}$. 