Math 126 - Fall 2014 - Homework 4
Hard copy due: Friday 10/03/2014 at 4:10pm

Problem 1. Let \( H : \mathbb{R} \setminus \{0\} \to \mathbb{R} \) be defined by
\[
H(x) = \begin{cases} 
1 & x > 0 \\
0 & x < 0. 
\end{cases}
\]
(i) Regarding \( H \) as a distribution via \( H(f) = \int_{\mathbb{R}} H(x)f(x)\,dx \), show that \( H' = \delta_0 \).
(ii) Show that
\[
H(x + t) - H(x - t) = \begin{cases} 
\text{sgn}(t) & -|t| \leq x \leq |t| \\
0 & \text{otherwise}, 
\end{cases}
\]
where \( \text{sgn}(t) = \begin{cases} 
1 & t > 0 \\
-1 & t < 0. 
\end{cases} \)

Problem 2. Find the solution to
\[
\begin{cases} 
u_{tt} - u_{xx} = 0 \quad (t, x) \in (0, \infty) \times (0, \infty) \\
u(0, x) = f(x) \quad x \in (0, \infty) \\
u_t(0, x) = g(x) \quad x \in (0, \infty), \\
u(t, 0) = 0 \quad t \in (0, \infty). 
\end{cases}
\]
where \( f(0) = g(0) = 0 \).

Hint: Extend \( u \) to be defined for all \( x \in \mathbb{R} \) in such a way that \( u \) solves the one dimensional wave equation.

Problem 3. Let \( f : \mathbb{R} \to \mathbb{R} \) be a function that is not linear, that is, \( f \) is not of the form \( f(z) = c_1z + c_2 \) for any \( c_1, c_2 \in \mathbb{R} \). For which values of \( a \in \mathbb{R} \) can you find a solution to \( u_{tt} - u_{xx} = 0 \) of the form \( u(t, x) = f(x - at) \)?

Problem 4. Fix \( -1 < v < 1 \) and define \( L : \mathbb{R}^4 \to \mathbb{R}^4 \) by
\[
L(t, x_1, x_2, x_3) = \left( \frac{t - vx_1}{\sqrt{1 - v^2}}, \frac{x_1 - vt}{\sqrt{1 - v^2}}, x_2, x_3 \right).
\]
Let \( u : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R} \). Show that \( \Box (u \circ L) = (\Box u) \circ L \).

Remark: Here \( \circ \) denotes composition, i.e. \( a \circ b(x) = a(b(x)) \).

Problem 5. Suppose \( u : (0, \infty) \times \mathbb{R}^3 \to \mathbb{R} \) solves the wave equation \( \Box u = 0 \) for \( (t, x) \in (0, \infty) \times \mathbb{R}^3 \). Suppose that \( u_t(0, x) \) and \( u(0, x) \) are both identically zero on \( \mathbb{R}^3 \setminus B_R(x_0) \) for some \( x_0 \in \mathbb{R}^3 \) and \( R > 0 \). Describe and draw the region of spacetime where it is possible for \( u \) to be non-zero.

Problem 6. Show that
\[
\frac{\partial}{\partial r} \left( \int_{B_r(x)} f(y)\,dy \right) = \int_{\partial B_r(x)} f(y)\,dS(y).
\]

Hint: First change variables so that the region of integration does not depend on \( r \).