Math 126 - Fall 2014 - Homework 4
Hard copy due: Friday 10/03/2014 at 4:10pm

Problem 1. Let $H : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be defined by

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0. \end{cases}$$

(i) Regarding H as a distribution via $H(f) = \int_{\mathbb{R}} H(x)f(x)\,dx$, show that $H' = \delta_0$.

(ii) Show that

$$H(x+t) - H(x-t) = \begin{cases} \operatorname{sgn}(t) & -|t| \le x \le |t| \\ 0 & \text{otherwise,} \end{cases} \qquad \text{where} \quad \operatorname{sgn}(t) = \begin{cases} 1 & t > 0 \\ -1 & t < 0. \end{cases}$$

Problem 2. Find the solution to

$$\begin{cases} u_{tt} - u_{xx} = 0 & (t,x) \in (0,\infty) \times (0,\infty) \\ u(0,x) = f(x) & x \in (0,\infty) \\ u_t(0,x) = g(x) & x \in (0,\infty), \\ u(t,0) = 0 & t \in (0,\infty). \end{cases}$$

where $f(0) = g(0) = 0$.

Hint: Extend u to be defined for all $x \in \mathbb{R}$ in such a way that u solves the one dimensional wave equation.

Problem 3. Let $f : \mathbb{R} \to \mathbb{R}$ be a function that is **not** linear, that is, f is **not** of the form $f(z) = c_1 z + c_2$ for any $c_1, c_2 \in \mathbb{R}$. For which values of $a \in \mathbb{R}$ can you find a solution to $u_{tt} - u_{xx} = 0$ of the form $u(t,x) = f(x - at)$?

Problem 4. Fix $-1 < v < 1$ and define $L : \mathbb{R}^4 \to \mathbb{R}^4$ by

$$L(t, x_1, x_2, x_3) = \left(\frac{t - vx_1}{\sqrt{1-v^2}}, \frac{x_1 - vt}{\sqrt{1-v^2}}, x_2, x_3 \right).$$

Let $u : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$. Show that $\Box(u \circ L) = (\Box u) \circ L$.

Remark: Here $\circ$ denotes composition, i.e. $a \circ b(x) = a(b(x))$.

Problem 5. Suppose $u : (0,\infty) \times \mathbb{R}^3 \to \mathbb{R}$ solves the wave equation $\Box u = 0$ for $(t,x) \in (0,\infty) \times \mathbb{R}^3$. Suppose that $u_t(0,x)$ and $u(0,x)$ are both identically zero on $\mathbb{R}^3 \backslash B_R(x_0)$ for some $x_0 \in \mathbb{R}^3$ and $R > 0$. Describe and draw the region of spacetime where it is possible for u to be non-zero.

Problem 6. Show that

$$\frac{\partial}{\partial r}\left(\int_{B_r(x)} f(y)\,dy \right) = \int_{\partial B_r(x)} f(y)\,dS(y).$$

Hint: First change variables so that the region of integration does not depend on r.