Math 126 - Fall 2014 - Homework 3 - Solutions

Problem 1. Let \(a > 0\). Write an integral formula for the solution to

\[
\begin{align*}
\{ & \quad u_t - \Delta u + au = 0 \quad \text{for} \quad (t, x) \in (0, \infty) \times \mathbb{R}^d \\
& \quad u(0, x) = f(x) \quad \text{for} \quad x \in \mathbb{R}^d.
\end{align*}
\]

If \(v(t, x)\) solves \(v_t - \Delta v = 0\) with \(v(0, x) = f(x)\) then \(u(t, x) = e^{-at}v(t, x)\) solves \(u_t - \Delta u + au = 0\) with \(u(0, x) = f(x)\). So

\[
u(t, x) = (4\pi t)^{-d/2} e^{-at} \int_{\mathbb{R}^d} e^{-|x-y|^2/4t} f(y) \, dy.
\]

Problem 2. Let \(v \in \mathbb{R}^d\). Write an integral formula for the solution to

\[
\begin{align*}
\{ & \quad u_t - \Delta u + v \cdot \nabla u = 0 \quad \text{for} \quad (t, x) \in (0, \infty) \times \mathbb{R}^d \\
& \quad u(0, x) = f(x) \quad \text{for} \quad x \in \mathbb{R}^d.
\end{align*}
\]

If \(w(t, x)\) solves \(w_t - \Delta w = 0\) with \(w(0, x) = f(x)\) then \(u(t, x) = w(t, x-tv)\) solves \(u_t - \Delta u + v \cdot \nabla u = 0\) with \(u(0, x) = f(x)\). So

\[
u(t, x) = (4\pi t)^{-d/2} \int_{\mathbb{R}^d} e^{-|x-tv-y|^2/4t} f(y) \, dy.
\]

Problem 3. Suppose \(u\) is a solution to

\[
\begin{align*}
\{ & \quad u_t - \Delta u = 0 \quad \text{for} \quad (t, x) \in (0, \infty) \times \mathbb{R}^d \\
& \quad u(0, x) = f(x) \quad \text{for} \quad x \in \mathbb{R}^d.
\end{align*}
\]

Show that if \(f(x) = f(-x)\) for all \(x \in \mathbb{R}^d\), then \(u(t, x) = u(t, -x)\) for all \(x \in \mathbb{R}^d\) and all \(t > 0\).

\[
u(t, -x) = (4\pi t)^{-d/2} \int_{\mathbb{R}^d} e^{-|x-y|^2/4t} f(y) \, dy
\]
\[
= (4\pi t)^{-d/2} \int_{\mathbb{R}^d} e^{-|x+z|^2/4t} f(-z) \, dz \quad (z = -y)
\]
\[
= (4\pi t)^{-d/2} \int_{\mathbb{R}^d} e^{-|x-z|^2/4t} f(z) \, dz \quad (f(z) = f(-z))
\]
\[
= u(t, x).
\]

Problem 4. Consider the heat equation \(u_t - u_{xx} = 0\) in dimension \(d = 1\). Fix \(a \in \mathbb{R}\setminus\{0\}\). Find a solution of the form \(u(t, x) = f(x - at)\) for some function \(f\).

If \(u(t, x) = f(x - at)\) then \(u_t - u_{xx} = -af' - f''\). A solution to \(-af' - f'' = 0\) is given by \(f(z) = -\frac{1}{a} e^{-az}\). Thus \(u(t, x) = -\frac{1}{a} e^{-a(x-at)}\) is a solution of this form.

Problem 5. Show that

\[
(4\pi t)^{-d/2} \int_{\mathbb{R}^d} e^{-|x|^2/4t} \, dx = 1 \quad \text{for all} \quad t > 0.
\]
Changing variables via $y = \frac{x}{2\sqrt{t}}$, $dy = (2\sqrt{t})^{-d} dx = (4t)^{-d/2} dx$, we have

$$
(4\pi t)^{-d/2} \int_{\mathbb{R}^d} e^{-|x|^2/4t} \, dx = \pi^{-d/2} \int_{\mathbb{R}^d} e^{-|y|^2} \, dy
$$

$$
= \pi^{d/2} \left( \prod_{j=1}^d \int_{\mathbb{R}} e^{-y_j^2} \, dy_j \right)
$$

Thus the problem reduces to showing

$$
\left( \int_{\mathbb{R}} e^{-x^2} \, dx \right)^2 = \pi.
$$

To this end we use polar coordinates to write

$$
\left( \int_{\mathbb{R}} e^{-x^2} \, dx \right)^2 = \int_{\mathbb{R}^2} e^{-(x^2+y^2)} \, dx \, dy
$$

$$
= \int_0^{2\pi} \int_0^\infty e^{-r^2} r \, dr \, d\theta
$$

$$
= 2\pi \int_0^\infty \frac{1}{2} e^{-u} \, du \quad (u = r^2)
$$

$$
= \pi.
$$

**Problem 6.** In class we proved that if $v_t - \Delta v = 0$, we have

$$
v(0,0) = \frac{1}{4\pi} \int \int \int_{E_r(0,0)} v(s,y) \frac{|y|^2}{s^2} \, dy \, ds.
$$

Use (*) to deduce that if $u_t - \Delta u = 0$, we have

$$
u(0,0) = \frac{1}{4\pi} \int \int \int_{E_r(0,0)} u(s,y) \frac{|x-y|^2}{(t-s)^2} \, dy \, ds.
$$

Given a solution $u$ and a point $(t,x)$ let $v(s,y) = u(t+s,x+y)$. Then $v_t - \Delta u = 0$, and

$$
u(t,x) = v(0,0) = \frac{1}{4\pi} \int \int \int_{E_r(0,0)} v(s,y) \frac{|y|^2}{s^2} \, dy \, ds
$$

$$
= \frac{1}{4\pi} \int \int \int_{E_r(0,0)} u(t+s,x+y) \frac{|y|^2}{s^2} \, dy \, ds
$$

$$
= \frac{1}{4\pi} \int \int \int_{E_r(t,x)} u(s,y) \frac{|x-y|^2}{(t-s)^2} \, dy \, ds.
$$

**Problem 7.** Suppose $u : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ is a smooth solution to $u_t - u_{xx} = 0$ such that $u$ and its derivatives are integrable. Show that $\frac{d}{dt} \int_{\mathbb{R}} u(t,x) \, dx \equiv 0$.

$$
\frac{d}{dt} \int_{-\infty}^{\infty} u(t,x) \, dx = \int_{-\infty}^{\infty} u_t(t,x) \, dx = \int_{-\infty}^{\infty} u_{xx}(t,x) \, dx = [u_x]_{-\infty}^{\infty} = 0.
$$