Problem 1. Suppose $v : \mathbb{R}^2 \to \mathbb{R}$ solves $\Delta v = 0$ on $B_1(0)$. Let $\theta \in (0, 2\pi)$ and define $M : B_1(0) \to B_1(0)$ by $M(x, y) = (x \cos \theta - y \sin \theta, x \sin \theta + y \cos \theta)$. Define $u : \mathbb{R}^2 \to \mathbb{R}$ by $u(x, y) = v(M(x, y))$. Show that $u$ solves $\Delta u = 0$ on $B_1(0)$.

Problem 2. Find a (non-constant) solution to $\Delta u = 0$ on $\mathbb{R}^2 \setminus \{0\}$.

*Hint:* Mimic the computations carried out in class for dimensions $d \geq 3$.

*Remark:* When I say “find” a solution, I don’t mean find it in the textbook. I mean for you to look for a solution the way we did in class.

Problem 3. Let $g \in C^\infty_0$. Write down an integral formula for the solution to

$$
\begin{cases}
-\Delta u = 0 & \text{on } B_1(0) \subset \mathbb{R}^3 \\
u = g & \text{on } \partial B_1(0).
\end{cases}
$$

*Hint:* Use the Green’s function for $B_1(0)$ that we computed in class.

Problem 4. Compute the Green’s function for the interval $(0, \ell) \subset \mathbb{R}$.

*Hint:* For the fundamental solution in dimension $d = 1$, look back at Homework 1.

Problem 5*. Compute the Green’s function for

$$
\Omega := \{x \in \mathbb{R}^3 : x_2, x_3 > 0\}.
$$

*Hint:* Revisit the computation of the Green’s function for the half-space $\mathbb{R}^d_+$. 

Problem 6*. This problem will walk you through a proof of the following result:

Suppose $u : \mathbb{R}^2 \to \mathbb{R}$ satisfies $\Delta u = 0$ on $\mathbb{R}^2$. Suppose further that

$$
\text{there exists } C > 0 \text{ such that } |u(x)| \leq C \text{ for all } x \in \mathbb{R}^2. \quad (\ast)
$$

Then $u$ is constant.

(i) Let $R > 0$ and $0 < \delta < R$. Denote $B = B_R(0, 0)$ and $B' = B_R(\delta, 0)$. Show that

$$
\text{Area}(B \setminus B' \cup B' \setminus B) \leq 4\delta R.
$$

*Hint:* You can actually compute the area exactly, but it is simpler to draw a picture and make a geometric argument.

(ii) Show that $|u(0, 0) - u(\delta, 0)| \leq \frac{4C\delta}{\pi R}$.

*Hint:* Use the mean value property, $(\ast)$, and (i).

One can then send $R \to \infty$ in (ii) to conclude that $u(0, 0) = u(\delta, 0)$. These arguments show that $u(0, 0) = u(x, 0)$ for any $x \in \mathbb{R}$. Arguing similarly, one can show $u(x, 0) = u(x, y)$ for any $x, y \in \mathbb{R}$. One concludes that $u(x, y) \equiv u(0, 0)$, that is, $u$ is constant.