Math 126 - Fall 2014 - Homework 2
Hard copy due: Wednesday 09/17/2014 at 4:10pm

Problem 1. Suppose $v : \mathbb{R}^2 \to \mathbb{R}$ solves $\Delta v = 0$ on $B_1(0)$. Let $\theta \in (0, 2\pi)$ and define

$$M : B_1(0) \to B_1(0) \quad \text{by} \quad M(x, y) = (x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta).$$

Define $u : \mathbb{R}^2 \to \mathbb{R}$ by $u(x, y) = v(M(x, y))$. Show that u solves $\Delta u = 0$ on $B_1(0)$.

Problem 2. Find a (non-constant) solution to $\Delta u = 0$ on $\mathbb{R}^2 \backslash \{0\}$.

Hint: Mimic the computations carried out in class for dimensions $d \geq 3$.

Remark: When I say "find" a solution, I don't mean find it in the textbook. I mean for you to look for a solution the way we did in class.

Problem 3. Let $g \in C_c^\infty$. Write down an integral formula for the solution to

$$\begin{cases} -\Delta u = 0 & \text{on } B_1(0) \subset \mathbb{R}^3 \\ u = g & \text{on } \partial B_1(0). \end{cases}$$

Hint: Use the Green's function for $B_1(0)$ that we computed in class.

Problem 4. Compute the Green's function for the interval $(0, \ell) \subset \mathbb{R}$.

Hint: For the fundamental solution in dimension $d = 1$, look back at Homework 1.

Problem 5*. Compute the Green's function for

$$\Omega := \{x \in \mathbb{R}^3 : x_2, x_3 > 0\}.$$

Hint: Revisit the computation of the Green's function for the half-space $\mathbb{R}^d_+$.

Problem 6*. This problem will walk you through a proof of the following result:

Suppose $u : \mathbb{R}^2 \to \mathbb{R}$ satisfies $\Delta u = 0$ on $\mathbb{R}^2$. Suppose further that

$$\textit{there exists } C > 0 \textit{ such that } |u(x)| \leq C \textit{ for all } x \in \mathbb{R}^2. \quad (*)$$

Then u is constant.

(i) Let $R > 0$ and $0 < \delta < R$. Denote $B = B_R(0, 0)$ and $B' = B_R(\delta, 0)$. Show that

$$\text{Area}\big(B \backslash B' \cup B' \backslash B\big) \leq 4\delta R.$$

Hint: You can actually compute the area exactly, but it is simpler to draw a picture and make a geometric argument.

(ii) Show that $|u(0, 0) - u(\delta, 0)| \leq \frac{4C\delta}{\pi R}$.

Hint: Use the mean value property, $(*)$, and (i).

One can then send $R \to \infty$ in (ii) to conclude that $u(0, 0) = u(\delta, 0)$. These arguments show that $u(0, 0) = u(x, 0)$ for any $x \in \mathbb{R}$. Arguing similarly, one can show $u(x, 0) = u(x, y)$ for any $x, y \in \mathbb{R}$. One concludes that $u(x, y) \equiv u(0, 0)$, that is, u is constant.