Problem 1. Let $Q, Q'$ be hermitian operators on $L^2$ and let $\Psi \in L^2$. Show that

$$E([Q, Q']) := \langle \Psi, [Q, Q'] \Psi \rangle$$

is pure imaginary, where $[Q, Q']$ is the commutator defined by $[Q, Q'] = QQ' - Q'Q$.

$$E([Q, Q']) = \langle \Psi, [Q, Q'] \Psi \rangle = \langle \Psi, QQ' \Psi \rangle - \langle \Psi, Q'Q \Psi \rangle$$

($Q, Q'$ hermitian)

$$= \langle \Psi, QQ' \Psi \rangle - \langle \Psi, Q'Q \Psi \rangle$$

$$= -\langle \Psi, [Q, Q'] \Psi \rangle = -E([Q, Q']) ,$$

thus $E([Q, Q'])$ is pure imaginary.

Problem 2. Recall the formula for polar coordinates in $\mathbb{R}^2$: $(x, y) = (r \cos \theta, r \sin \theta)$.

Write $u(x, y) = v(r, \theta)$. Show that

$$\Delta u = r \frac{\partial^2 v}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 v}{\partial \theta^2}.$$

Since $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(y/x)$ we have

$$\frac{\partial r}{\partial x} = \cos \theta, \quad \frac{\partial r}{\partial y} = \sin \theta, \quad \frac{\partial \theta}{\partial x} = -\frac{\sin \theta}{r}, \quad \frac{\partial \theta}{\partial y} = \frac{\cos \theta}{r}.$$ 

Thus

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial r} \cos \theta - \frac{\partial v}{\partial \theta} \sin \frac{\theta}{r}.$$ 

So

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial r^2} \cos^2 \theta + \frac{\sin^2 \theta}{r} \frac{\partial v}{\partial r} + \frac{\sin^2 \theta}{r^2} \frac{\partial^2 v}{\partial \theta^2} + \frac{\partial v}{\partial r} \frac{2 \cos \theta \sin \theta}{r^2}. $$

Similarly

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial r} \sin \theta + \frac{\partial v}{\partial \theta} \frac{\cos \theta}{r}. $$

So

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial r^2} \cos^2 \theta + \frac{\cos^2 \theta}{r} \frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial \theta^2} \frac{\sin^2 \theta}{r^2} + \frac{\partial v}{\partial r} \frac{2 \sin \theta \cos \theta}{r^2}. $$

Thus

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{1}{r^2} \frac{\partial^2 v}{\partial \theta^2}.$$ 

Problem 3. Consider the Schrödinger equation on $\mathbb{R} \times \mathbb{R}^d$ with a real-valued potential:

$$i\hbar \frac{\partial}{\partial t} \Psi = -\frac{\hbar^2}{2m} \Delta \Psi + V(x) \Psi.$$ 

Suppose $\Psi$ is a normalized solution and define $\tilde{\Psi}$ by $\tilde{\Psi}(t, x) = e^{i\theta} \Psi(t, x)$ for some $\theta \in \mathbb{R}$.

(i) Note that $\tilde{\Psi}$ is a normalized solution to the Schrödinger equation.

(ii) Let $Q$ be any observable. Show that the expected value of $Q$ in state $\Psi$ is equal to the expected value of $Q$ in state $\tilde{\Psi}$.

Part (i) is clear (since the $e^{i\theta}$ factors out of each side of the equation). For part (ii), we have the following:

$$\langle \tilde{\Psi}, Q \tilde{\Psi} \rangle = \int e^{i\theta} \Psi Q [e^{i\theta} \Psi] = \int \Psi e^{-i\theta} e^{i\theta} Q \Psi = \langle \Psi, Q \Psi \rangle.$$
Problem 4. Use the power series method to solve the eigenvalue problem for the harmonic oscillator in one dimension:

\[-\frac{\hbar^2}{2m}q'' + \frac{1}{2}kx^2q = \lambda q.\]

Let us introduce \(z = (\frac{km}{\hbar})^{1/4}x\) and write \(w(z) = q(x)\). Then the ODE becomes

\[w'' = (z^2 - \mu)w, \quad \mu = \frac{2m^{1/2}\lambda}{\hbar k^{1/2}}.\]

For large \(z\) this ODE is approximately \(w'' = z^2w\). Now, \(a(z) := e^{-z^2/2}\) is an approximate solution of the ODE \(w'' = z^2w\) (for large \(z\)). We then write \(w(z) = a(z)r(z)\) and derive the following ODE for \(r\):

\[r'' - 2zr' + (\mu - 1)r = 0.\]

We look for a solution of the form \(r(z) = \sum_{j=0}^{\infty} c_j z^j\). The ODE then turns into the following recurrence relation for the coefficients:

\[c_{j+2} = \frac{2j-(\mu-1)}{(j+1)(j+2)} c_j.\]

As suggested, we only look for solutions for which the series terminates. This means we only consider \(\mu = \mu_n\) such that

\[\mu = 1 + 2n \quad \text{for some } n \in \{0\} \cup \mathbb{N}.\]

Let us also not concern ourselves with describing the eigenfunctions, but rather unravel the constants to figure out the eigenvalues \(\lambda_n\). In particular, we use the definition of \(\mu\) to see that the eigenvalues are

\[\lambda_n = \frac{\hbar k^{1/2}}{m^{1/4}}(n + \frac{1}{2}).\]

Problem 5. Let \(F : \mathbb{R}^3 \to \mathbb{R}^3\). Prove the identity

\[\nabla \times (\nabla \times F) = \nabla (\nabla \cdot F) - \Delta F,\]

where \(\nabla\) denotes gradient, \(\nabla \times\) denotes curl, \(\nabla \cdot\) denotes divergence, and \(\Delta F = (\Delta F_1, \Delta F_2, \Delta F_3)\).

The identity above is a vector identity, so there are really three identities to prove. Let me just write down the first component. On the left-hand side we have

\[\partial_2(\nabla \times F)_3 - \partial_3(\nabla \times F)_2 = \partial_2(\partial_1 F_2 - \partial_2 F_1) + \partial_3(\partial_1 F_3 - \partial_3 F_1) = \partial_2 \partial_1 F_2 - \partial_2 \partial_2 F_1 + \partial_3 \partial_1 F_3 - \partial_3 \partial_3 F_1.\]

On the right-hand side we have

\[\partial_1(\partial_1 F_1 + \partial_2 F_2 + \partial_3 F_3) - (\partial_1 \partial_1 F_1 + \partial_2 \partial_2 F_1 + \partial_3 \partial_3 F_1) = \partial_1 \partial_2 F_2 - \partial_1 \partial_3 F_3 - \partial_2 \partial_2 F_1 - \partial_3 \partial_3 F_1.\]

Using equality of mixed partials we conclude the two sides are equal.

The other components are similar.

Problem 6. (optional exercise) Recall that for a state \(\Psi \in L^2\) and an operator \(Q\) we have

\[E(Q) := \langle \Psi, Q\Psi \rangle, \quad \sigma_Q^2 := \| (Q - E(Q))\Psi \|^2,\]

where \(\langle f, g \rangle = \int \overline{f(x)}g(x)\) and \(\|f\|^2 = \langle f, f \rangle\). Show that for any state \(\Psi \in L^2\) and any two observables \(Q, Q'\) we have the uncertainty principle:

\[\sigma_Q^2 \sigma_{Q'}^2 \geq \frac{1}{2}(E([Q,Q'])).\]
Recalling the definitions of $\sigma^2_Q$ and $\sigma^2_{Q'}$, using the Cauchy–Schwarz inequality, and using the fact that $|z|^2 \geq (\text{Im} z)^2$, we get

$$\sigma^2_Q \sigma^2_{Q'} \geq |\langle (Q - E(Q))\Psi, (Q' - E(Q'))\Psi \rangle|^2 \geq (\text{Im} \langle [Q - E(Q)]\Psi, [Q' - E(Q')]\Psi \rangle)^2. \quad (*)$$

Let’s simplify the formulas a little bit by writing $\lambda = E(Q)$ and $\lambda' = E(Q')$. Recall that $\lambda, \lambda'$ are real since $Q, Q'$ are hermitian.

We use the formula $\text{Im} z = \frac{1}{2i}(z - \overline{z})$ and recall $\langle f, g \rangle = \overline{\langle g, f \rangle}$. We can expand things out and use that $Q, Q'$ are hermitian to find:

$$\text{Im} \langle (Q - \lambda)\Psi, (Q' - \lambda')\Psi \rangle = \frac{1}{2i} \left( \langle Q\Psi - \lambda\Psi, Q'\Psi - \lambda'\Psi \rangle - \langle Q'\Psi - \lambda'\Psi, Q\Psi - \lambda\Psi \rangle \right)$$

$$= \frac{1}{2i} \left[ \langle \Psi, QQ'\Psi \rangle - \lambda \langle \Psi, Q'\Psi \rangle - \lambda' \langle Q\Psi, \Psi \rangle + \lambda\lambda' \langle \Psi, \Psi \rangle \right.$$

$$\left. - (\langle \Psi, Q'Q\Psi \rangle - \lambda' \langle \Psi, Q'\Psi \rangle - \lambda \langle Q'\Psi, \Psi \rangle + \lambda\lambda' \langle \Psi, \Psi \rangle) \right]$$

$$= \frac{1}{2i} \langle \Psi, (QQ' - Q'Q)\Psi \rangle = -\frac{i}{2} E([Q, Q']).$$

The result follows.