Math 126 - Fall 2014 - Homework 10 - Solutions

Problem 1. Consider the following boundary-value/initial-value problem for the heat equation:

\[(*) \quad \begin{cases} \frac{u(t_{n+1}, x_j) - u(t_{n-1}, x_j)}{2\Delta t} = \frac{u(t_n, x_{j+1}) + u(t_n, x_{j-1}) - 2u(t_n, x_j)}{\Delta x^2} & (j = 1, \ldots, J - 1), \\ u(t_0, x_j) = f(x_j) & (j = 0), \\ u(t_{n-1}, x_j) = u(t_n, x_j) = 0 & (n = 0, \ldots, N). \end{cases}\]

Let \( J \in \mathbb{N} \) and \( \Delta x = \frac{1}{J} \). Let \( \Delta t > 0 \) and define \( r = \frac{\Delta t}{\Delta x^2} \).

Define \( \{x_j\}_{j=0}^J \) by \( x_j = j\Delta x \) and \( \{t_n\}_{n=-1}^N \) by \( t_n = n\Delta t \).

(i) Show that the following scheme for \((*)\) is stable for all \( r \):

\[
\begin{align*}
\frac{u(t_{n+1}, x_j) - u(t_{n-1}, x_j)}{2\Delta t} &= \frac{u(t_n, x_{j+1}) + u(t_n, x_{j-1}) - 2u(t_n, x_j)}{\Delta x^2} \quad (j = 1, \ldots, J - 1), \\
u(t_0, x_j) &= f(x_j) \quad (j = 0), \\
u(t_{n-1}, x_j) &= u(t_n, x_j) = 0 \quad (n = 0, \ldots, N). 
\end{align*}
\]

Let’s simplify our lives and write \( u^n_j = u(t_n, x_j) \). Rearranging the scheme gives

\[(1 + 2r)u^{n+1}_j = (1 - 2r)u^{n-1}_j + 2ru^n_{j+1} + u^n_{j-1} \cdot \]

Consider a separated solution \( u^n_j = p_k(t_n)q_k(x_j) \), where \( q_k(x_j) = e^{i\kappa \pi x_j} \). Then

\[
(1+2r)p_k(t_{n+1}) - (1-2r)p_k(t_{n-1}) = \frac{q_k(x_{j+1}) + q_k(x_{j-1})}{q_k(x_j)} = \lambda_k
\]

for some \( \lambda_k \). In particular the equation for \( q_k \)’s gives

\[
\lambda_k = 2 \cos(k\pi \Delta x).
\]

For the \( p_k \)’s we make the ansatz \( p_k(t_n) = \mu_k^n p_k(t_0) \) for some \( \mu_k \), and we wish to show \( |\mu_k| \leq 1 \).

This leads to

\[(1 + 2r)\mu_k^2 - 2r\lambda_k \mu_k - (1 - 2r) = 0.\]

Solving this quadratic equation gives

\[
\mu_k = \frac{r\lambda_k \pm \sqrt{r^2 - 4r^2(4-\lambda_k^2)}}{2r}.
\]

First consider the case \( r^2(4 - \lambda_k^2) \leq 1 \). Then the square root above is real and bounded in magnitude by \( 1 \). Using that \( -2 \leq \lambda_k \leq 2 \), we find

\[-1 = \frac{-2r - 1}{1 + 2r} \leq \mu_k \leq \frac{1 + 2r}{1 + 2r} = 1.\]

If \( r^2(4 - \lambda_k^2) > 1 \) then the square root above is purely imaginary. The magnitude squared of \( \mu_k \) is thus

\[
|\mu_k|^2 = \frac{r^2\lambda_k^2 + r^2(4 - \lambda_k^2) - 1}{(1 + 2r)^2} = \frac{2r - 1}{2r + 1} \leq 1.
\]

(ii) Show that the following scheme for \((*)\) is unstable for all \( r \):

\[
\begin{align*}
\frac{u(t_{n+1}, x_j) - u(t_{n-1}, x_j)}{2\Delta t} &= \frac{u(t_n, x_{j+1}) - 2u(t_n, x_j) + u(t_n, x_{j-1})}{\Delta x^2} \quad (j = 1, \ldots, J - 1), \\
u(t_0, x_j) &= f(x_j) \quad (j = 0), \\
u(t_{n-1}, x_j) &= u(t_n, x_j) = 0 \quad (n = 0, \ldots, N). 
\end{align*}
\]
As before we write \( u(t_n, x_j) = u^n_j \) and rearrange:

\[
 u^{n+1}_j = u^n_j + 2r[u^n_{j+1} - 2u^n_j + u^n_{j-1}] .
\]

Looking for a separated solution \( u^n_j = p_k(t_n)q_k(x_j) \) as above leads to

\[
 \frac{p_k(t_{n+1}) - p_k(t_{n-1})}{2rp_k(t_n)} = \frac{q_k(x_{j+1})+q_k(x_{j-1})}{q_k(x_j)} - 2 = \lambda_k .
\]

Thus

\[
 \lambda_k = 2 \cos(k\pi \Delta x) - 2 \leq 0 
\]

and plugging in the ansatz \( p_k(t_n) = \mu_k^n p_k(t_0) \) yields

\[
 \mu_k^2 - 2r\lambda_k \mu_k - 1 = 0 .
\]

Solving this gives

\[
 \mu_k = r\lambda_k \pm \sqrt{1 + r^2\lambda_k^2} 
\]

But now we notice that since \( \lambda_k \leq 0 \) we can show

\[
 r\lambda_k - \sqrt{1 + r^2\lambda_k^2} < -1 ,
\]

which implies the scheme is unstable.

**Problem 2.** Let \( \Omega = (0, 1) \times (0, 1) \subset \mathbb{R}^2 \). Suppose \( u \) is the solution to Poisson’s equation

\[
 \begin{cases}
 -\Delta u = 4 & \text{in } \Omega \\
 u = 0 & \text{on } \partial \Omega .
\end{cases}
\]

Partition \( \Omega \) into four triangles \( \{T_j\}_{j=1}^4 \) by drawing its diagonals. Let \( \varphi : \overline{\Omega} \to \mathbb{R} \) be the piecewise linear “test function” defined by \( \varphi(x, y) = a_j + b_j x + c_j y \) for \( (x, y) \in T_j \), where \( \{a_j, b_j, c_j\} \) are determined by imposing

\[
 \varphi \left( \frac{1}{2}, \frac{1}{2} \right) = 1 , \quad \text{and} \quad \varphi = 0 \text{ on } \partial \Omega .
\]

Use the finite element method (with the single test function \( \varphi \)) to approximate \( u(\frac{1}{2}, \frac{1}{2}) \).

Let \( T_1 \) be the triangle containing \( (0, 0) \) and \( (0, 1) \). Let \( T_2 \) be the triangle containing \( (0, 1) \) and \( (1, 1) \). Let \( T_3 \) be the triangle containing \( (1, 1) \) and \( (1, 0) \). Let \( T_4 \) be the triangle containing \( (1, 0) \) and \( (0, 0) \).

Let \( \varphi = \varphi_j \) on triangle \( T_j \). Then we can solve for \( a_j, b_j, c_j \) and find

\[
 \varphi_1(x, y) = 2x , \quad \varphi_2(x, y) = 2 - 2y , \quad \varphi_3(x, y) = 2 - 2x , \quad \varphi_4(x, y) = 2y .
\]

We wish to write \( u = \alpha \varphi \), where \( \alpha \) is chosen so that

\[
 \alpha \int_{\Omega} |\nabla \varphi|^2 = \int_{\Omega} 4\varphi . \tag{\ast}
\]

To compute these integrals, break \( \Omega \) into the union of the \( T_j \)'s. On each \( T_j \) we have \( |\nabla \varphi_j|^2 \equiv 4 \), and the total area of \( \Omega \) is one. So the left-hand side equals \( 4\alpha \). It remains to compute the right-hand side. By symmetry, it suffices to compute, say \( \int_{T_4} \varphi_4 \) and multiply our answer by 4. We have

\[
 \int_{T_4} \varphi_4 = \int_{0}^{1/2} 2y \int_{y}^{1-y} dx dy = \frac{1}{12},
\]

so the right-hand side of (\ast) is equal to \( 4/3 \). Thus we should take \( 4\alpha = 4/3 \), i.e. \( \alpha = 1/3 \). We then approximate \( u(\frac{1}{2}, \frac{1}{2}) = \alpha \varphi(\frac{1}{2}, \frac{1}{2}) = \alpha = \frac{1}{3} \).
**Problem 3.** Suppose \( V : \mathbb{R}^3 \to \mathbb{R} \) is a radial function, that is, \( V(x) = v(|x|) \) for some \( v : \mathbb{R} \to \mathbb{R} \). Suppose \( (x(t), p(t)) \in \mathbb{R}^3 \times \mathbb{R}^3 \) solves
\[
\begin{align*}
\dot{x}(t) &= \frac{1}{m} p(t), \quad x(0) = x_0, \\
\dot{p}(t) &= -\nabla V(x(t)), \quad p(0) = p_0.
\end{align*}
\]
Define \( L(t) = x(t) \times p(t) \), where \( \times \) is the cross product. Show that \( L(t) \equiv x_0 \times p_0 \).

Write \( V(x) = v(r(x)) \) where \( r(x) = |x| \). Then \( \nabla V(x) = v'(|x|) \frac{x}{|x|} \). Then
\[
\begin{align*}
\partial_t L(t) &= \dot{x}(t) \times p(t) + x(t) \times \dot{p}(t) \\
&= \frac{1}{m} p(t) \times p(t) - x(t) \times \nabla V(x(t)) \\
&= \frac{1}{m} p(t) \times p(t) - [v'(|x(t)|)] |x(t)| x(t) \\
&= 0 + 0.
\end{align*}
\]

**Problem 4.** Suppose \( \Psi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C} \) is a smooth decaying solution to the Schrödinger equation
\[
i\hbar \Psi_t = -\frac{\hbar^2}{2m} \Delta \Psi + V(x) \Psi
\]
for some \( V : \mathbb{R}^d \to \mathbb{R} \). Show that \( \frac{d}{dt} \int_{\mathbb{R}^d} |\Psi(t, x)|^2 \, dx = 0 \).

\[
\begin{align*}
\partial_t \int \Psi \overline{\Psi} &= \int \Psi_t \overline{\Psi} + \Psi \overline{\Psi}_t \\
&= \int 2 \text{Re} \left[ \overline{\Psi} \Psi_t \right] \\
&= \int 2 \text{Re} \left[ \overline{\Psi} \left( \frac{i\hbar}{2m} \Delta \Psi - \frac{i}{\hbar} V \Psi \right) \right] \\
&= -\int 2 \text{Im} \left[ \overline{\Psi} \left( \frac{\hbar}{2m} \Delta \Psi - \frac{1}{\hbar} V \Psi \right) \right] \\
&= -2 \left[ -\frac{\hbar}{2m} |\nabla \Psi|^2 - \frac{1}{\hbar} V |\Psi|^2 \right] \\
&= 0.
\end{align*}
\]

**Problem 5.** Show that the operators \( x \) and \( p \) are hermitian with respect to the \( L^2 \) inner product, where
\[
[x\Psi](x) := x \Psi(x) \quad \text{and} \quad [p\Psi](x) := -i\hbar \nabla \Psi(x).
\]
First, since the components of \( x \) are real,
\[
\int \overline{\Psi} x \Psi = \int \overline{x} \Psi \Psi.
\]
Next, using integration by parts in each component and using \( \overline{\imath} = i \),
\[
\begin{align*}
\int \overline{\Psi} [p\Psi] &= \int \overline{\Psi} [-i\hbar \nabla \Psi] = -\int \overline{\nabla \Psi} (-i\hbar) \Psi = \int \overline{\nabla \Psi} \Psi = \int \overline{[p\Psi]} \Psi.
\end{align*}
\]