Math 110 Fall 2019 - Quiz 10 - BANK

date - time **Name:**

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(1) (5 points) Let A be a 2×2 matrix with entries in $\mathbb{R}$ and let $x,\ y \in \mathbb{R}^2$ (column vectors). Prove or disprove: $\langle x, y \rangle = x^T A y$ defines an inner product.

(2) (5 points) Let e_1 and e_2 be the standard basis vectors in $\mathbb{R}^2$. Prove or disprove the following: There exists an inner product on $\mathbb{R}^2$ such that $\langle e_1, e_1 - e_2 \rangle = \langle e_2, e_1 - e_2 \rangle = 1$.

(3) (5 points) Let e_1 and e_2 be the standard basis vectors in $\mathbb{R}^2$. Prove or disprove the following: There exists an inner product on $\mathbb{R}^2$ such that $\langle e_1, e_1 \rangle = \langle e_1, e_2 \rangle = 1$.

(4) (5 points) Let T be a linear operator on an inner product space V. Prove that the operators $S = T + T^*$ and $U = T^* T$ are self-adjoint (meaning that $S^* = S$ and $U^* = U$).

(5) (5 points) Consider $V = P^1([0,1])$ with the inner product, $\langle f, g \rangle = \int_0^1 f(x) g(x) dx$. What is the orthogonal compliment of $S = \{x - 1\}$?

(6) (5 points) Let $V = \mathbb{R}^3$ with the standard inner product. What is the orthogonal compliment of $S = \{e_1 + e_3, e_1 - e_2\}$?

Solution to (3):

I claim the following is an inner product that satisfies the given conditions. Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Consider,

$$\langle x, y\rangle = x_1y_1 + x_1y_2 + x_2y_1 + 2x_2y_2$$

It is left to the reader to show that the above satisfies the conditions of the problem, linearity in the first component, and (conjugate) symmetry. We will show that the inner product satisfies positive definiteness. Let $x \neq 0$,

$$\begin{aligned}\langle x, x\rangle &= x_1^2 + 2x_1x_2 + 2x_2^2\\ &= x_1^2 + 2x_1x_2 + x_2^2 + x_2^2\\ &= (x_1 + x_2)^2 + x_2^2 > 0\end{aligned}$$

The last line is strictly greater that 0 because if the first term is 0 the second term is positive and vice versa.