§ Spin Structures on Vector bundles.

Overview: Let $\pi : E \to X$ be a real n-dimensional oriented riemannian vector bundle, using the riemannian metric we consider the bundle $P_O(E)$ of orthonormal frames in E (a principal $O(n)$ bundle):

$$\begin{tikzcd} O(n) \arrow[d] \\ P_O(E) \arrow[d] \\ X \end{tikzcd}$$

and choosing an orientation is equivalent to choosing a principal $SO(n)$ bundle $P_{SO}(E) \subset P_O(E)$ satisfying the compatibility condition:

$$\begin{tikzcd} SO(n) \arrow[r, hook, "i_0"] \arrow[d] & O(n) \arrow[d] \\ P_{SO}(E) \arrow[r, hook, "i"] \arrow[dr, "\pi'"'] & P_O(E) \arrow[d, "\pi"] \\ & X \end{tikzcd}$$

so $i(y\,s) = i(y) \cdot i_0(s)$ for $y \in P_{SO}(E)$, $s \in SO(n)$.

- Recall from week 2 that for $n \geq 3$ we have the universal covering homomorphism:

$$\xi_0 : \mathrm{Spin}(n) \longrightarrow SO(n)$$

with $\ker(\xi_0) \cong \mathbb{Z}/2\mathbb{Z}$. This motivates:

Definition: Suppose $n \geq 3$. A Spin structure on a real n-dim'l oriented riemannian vector bundle E is a principal $\mathrm{Spin}(n)$-bundle $P_{\mathrm{Spin}}(E)$ together with a two sheeted covering

$$\xi : P_{\mathrm{Spin}(n)}(E) \longrightarrow P_{SO(n)}(E)$$

Satisfying compatibility:

$$\begin{tikzcd}
 & \mathrm{Spin}(n) \arrow[r, "\xi_0"] \arrow[d] & SO(n) \arrow[d] \\
\mathbb{Z}/2\mathbb{Z} \arrow[ur] \arrow[r] & P_{Spin}(E) \arrow[r, "\xi"] \arrow[dr, "\pi''"'] & P_{SO}(E) \arrow[d, "\pi'"] \\
 & & X
\end{tikzcd}$$

$\left(\xi(pg) = \xi(p)\,\xi_0(g) \quad p \in P_{Spin}(E),\ g \in \mathrm{Spin}(n) \right)$

<u>We ask</u>: When does an oriented Riemannian vector bundle E admit a spin structure?

To answer this question we'll need a couple facts regarding classifying spaces and a theorem from obstruction theory.

<u>Definition</u>: A <u>Classifying Space</u> for a group G is a connected topological space BG, together with a principal G-bundle $EG \longrightarrow BG$ such that the following is true:

For any compact Hausdorff space X, there is a one-to-one correspondence between equivalence classes of principal G-bundles on X and homotopy classes of maps from X to BG, given by:

$$\begin{tikzcd}
f^*EG \arrow[r, "\hat{f}"] \arrow[d] & EG \arrow[d] \\
X \arrow[r, "f"] & BG
\end{tikzcd}$$

(recall the <u>vector bundle induced by f</u>.)

or: $[f^*EG] = [\eta] \longleftrightarrow [f]$ (every class of principal G-bundles $[\eta]$ can be realized this way*)

*Classifying spaces are big!

Theorem: For any Lie group, G there exists a classifying space BG.

- note: $B : G \longmapsto BG$ is a functor from the category of topological groups to the homotopy category of CW-complexes.

Fact: The functor B takes the exact sequence:

$$1 \to \mathbb{Z}_2 \to Spin(n) \to SO(n) \to 1$$

to the fibration: $\Downarrow B$

$$\mathbb{RP}^\infty \to B\,spin(n) \to BSO(n)$$

☆ Putting all of this together we turn the question of existence of a spin structure on a vector bundle $E \to X$ into a question regarding homotopy classes of maps from X to $BSO(n)$ and $B\,Spin(n)$:

$$\begin{tikzcd}
\mathbb{Z}_2 \arrow[r] & P_{Spin}(E) \arrow[r] & P_{SO}(E) \arrow[d, "\pi'"] \\
 & & X \arrow[ul, dashed, "\exists!"]
\end{tikzcd}
\iff
\begin{tikzcd}
\mathbb{RP}^\infty \arrow[r] & B\,Spin(n) \arrow[r] & BSO(n) \\
 & & X \arrow[ul, dashed, "\exists!"] \arrow[u, "f"]
\end{tikzcd}
\quad f \longleftrightarrow \boxed{E \to X}$$

(Existence of Spin Structure) (Existence of lift of f)

Finally, from obstruction theory in fibrations, the only obstruction to lifting f is an element of $H^2(X; \pi_1(\mathbb{RP}^\infty)) = H^2(X;\mathbb{Z}_2)$, it is the second Stiefel-Whitney class of $E \to X$ denoted $W_2(E)$,

(in particular)
Note: By the Naturality of the Stiefel-Whitney classes $(W_i(E_1) = f^* W_i(E_2))$ when:

$$\begin{tikzcd}
E_1 \arrow[r, "\hat{f}"] \arrow[d] & E_2 \arrow[d] \\
B_1 \arrow[r, "f"] & B_2
\end{tikzcd}$$

We observe: a spin structure on $E \to X$ exists $\iff f^* W_2(ESO(n)) = 0$.

$H^2(X;\mathbb{Z}_2) \ni$

- Why does all of that matter?
- why care about spin structures?

• Recall that, given a riemannian manifold (M, g) we can define a bundle of clifford algebras over M by:

$$Cl(T_x M) := \left(\sum_{r=0}^{\infty} \otimes^r T_x M\right) / I(T_x M)$$

where $I(T_x M)$ is generated by elements $\vec{v} \otimes \vec{v} + g(\vec{v}, \vec{v})$ for $\vec{v} \in T_x M$.

- The existence of a spin structure on TM will enable us to define a bundle of $Cl(T_x M)$-modules called the Spinor bundle of TM, and every spinor bundle can be decomposed into a direct sum of irreducible spinor bundles.
- This construction will allow the application of the notions presented in weeks 1–4 to the study of manifolds, in future weeks, we'll observe the fruitfulness of this approach.