

Spin Structures on Vector bundles.

Overview: Let $\pi: E \rightarrow X$ be a real n -dimensional oriented riemannian vector bundle, using the riemannian metric we consider the bundle $P_o(E)$ of orthonormal frames in E (a principal $O(n)$ bundle):

$$\begin{array}{c} O(n) \\ \downarrow \\ P_o(E) \\ \downarrow \\ X \end{array}$$

and choosing an orientation is equivalent to choosing a principal $SO(n)$ bundle $P_{so}(E) \subset P_o(E)$ satisfying the Compatibility Condition:

$$\begin{array}{ccc} SO(n) & \xrightarrow{i_o} & O(n) \\ \downarrow & & \downarrow \\ P_{so}(E) & \xrightarrow{i} & P_o(E) \\ & \searrow \pi' & \downarrow \pi \\ & & X \end{array}$$

$$\text{so } i(y \cdot s) = i(y) \cdot i_o(s) \text{ for } y \in P_{so}(E), s \in SO(n).$$

- Recall from week 2 that for $n \geq 3$ we have the universal covering homomorphism:

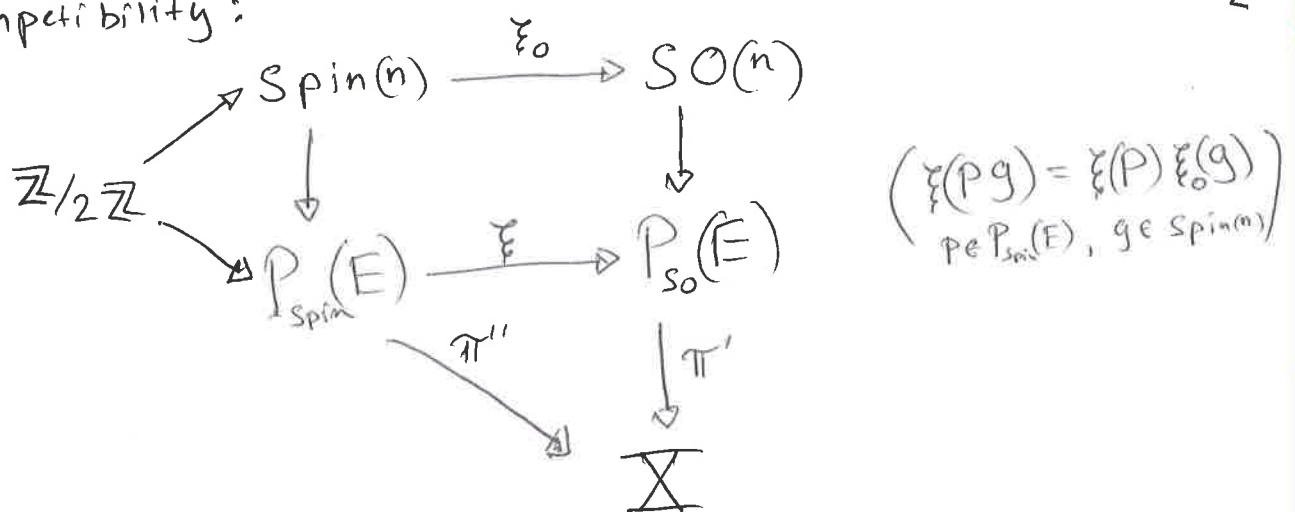
$$\xi_o: Spin(n) \rightarrow SO(n)$$

with $\ker(\xi_o) \cong \mathbb{Z}/2\mathbb{Z}$. This motivates:

Definition: Suppose $n \geq 3$. A Spin structure on a real n -dim'l oriented riemannian vector bundle E is a principal $Spin(n)$ -bundle $P_{spin}(E)$ together with a two sheeted covering

$$\xi: P_{spin}(E) \rightarrow P_{so}(E)$$

Satisfying compatibility:



We ask: When does an oriented Riemannian vector bundle E admit a spin structure?

to answer this question we'll need a couple facts regarding classifying spaces and a theorem from obstruction theory.

Definition: A classifying space for a group G is a connected topological space BG , together with a principal G -bundle $EG \rightarrow BG$ such that the following is true:

For any compact Hausdorff space X , there is a one-to-one correspondence between equivalence classes of principal G -bundles on X and homotopy classes of maps from X to BG , given by:

$$\begin{array}{ccc}
 f^*EG & \xrightarrow{\hat{f}} & EG \\
 \downarrow & & \downarrow \\
 X & \xrightarrow{f} & BG
 \end{array}$$

(Recall the vector bundle induced by f .)

or: $f^*EG = [\eta] \longleftrightarrow [f]$ (every $\checkmark$ principal G -bundles $[\eta]$ can be realized this way⁺)

class of

*Classifying spaces are big!

Theorem: For any Lie group, G there exists a classifying space BG .

- note: $B: G \mapsto BG$ is a functor from the category of topological groups to the homotopy category of CW-complexes.

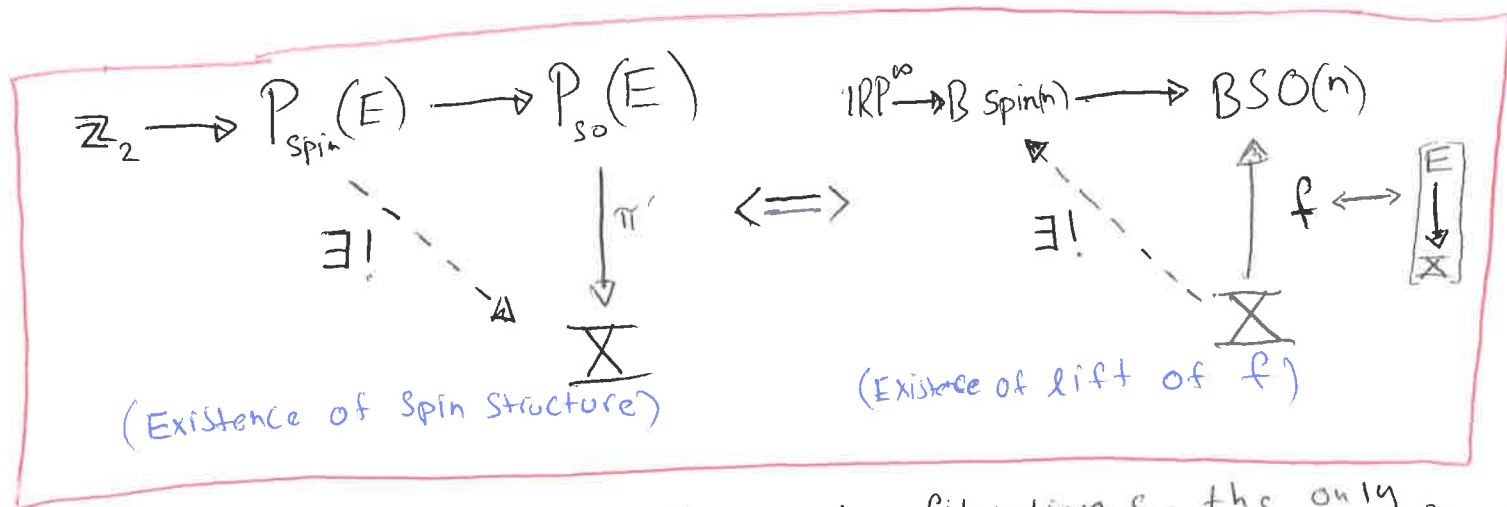
Fact: The functor B takes the exact sequence:

$$1 \rightarrow \mathbb{Z}_2 \rightarrow \text{Spin}(n) \rightarrow \text{SO}(n) \rightarrow 1$$

to the fibration: $\downarrow B$

$$\mathbb{R}P^\infty \rightarrow B\text{Spin}(n) \rightarrow B\text{SO}(n)$$

Putting all of this together we turn the question of existence of a spin structure on a vector bundle $E \rightarrow X$ into a question regarding homotopy classes of maps from X to $B\text{SO}(n)$ and $B\text{Spin}(n)$:



Finally, from obstruction theory in fibrations, the only obstruction to lifting f is an element of $H^2(X; \pi_1(\mathbb{R}P^\infty)) = H^2(X; \mathbb{Z}_2)$, it is the second Stiefel-Whitney class of $E \rightarrow X$ denoted $W_2(E)$,
(in particular)

Note: By the Naturality of the Stiefel-Whitney classes ($W_i(E_1) = f^* W_i(E_2)$)

When: ~~the map~~

$$\begin{array}{ccc} E_1 & \xrightarrow{\hat{f}} & E_2 \\ \downarrow & & \downarrow \\ B_1 & \xrightarrow{f} & B_2 \end{array}$$

$$H^2(X; \mathbb{Z}_2)$$

We observe: a spin structure on $E \rightarrow X$ exists $\iff f^* W_2(E\text{SO}(n)) = 0$.

- Why does all of that matter?
- Why care about spin structures?
- Recall that, given a Riemannian manifold (M, g) we can define a bundle of Clifford algebras over M by:

$$\mathcal{C}\ell(T_x M) := \left(\sum_{p=0}^{\infty} \otimes^p T_x M \right) / \mathcal{I}(T_x M)$$

Where $\mathcal{I}(T_x M)$ is generated by elements $\vec{v} \otimes \vec{v} + g(\vec{v}, \vec{v})$ for $\vec{v} \in T_x M$.

- The existence of a spin structure on TM will enable us to define a bundle of $\mathcal{C}\ell(T_x M)$ -modules called the Spinor bundle of TM , and every spinor bundle can be decomposed into a direct sum of irreducible spinor bundles.
- This construction will allow the application of the notions presented in weeks 1-4 to the study of manifolds, in future weeks, we'll observe the fruitfulness of this approach.