Representations of Clifford algebras

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February 13 2018

Abstract

We describe some representations of Clifford algebras, following *Spin Geometry*.

Warning: I am used to representation theory, so I may gloss over things that aren't actually obvious.

1 Basic definitions

Definition 1. Let K be a k-algebra.[1] A *K-representation* of the Clifford algebra $\mathfrak{Cl}(V, q)$ is a morphism of algebras

$$\rho : \mathfrak{Cl}(V, q) \to \hom_K(W, W)$$

where W is a K-algebra representation, and $\hom_K(W, W)$ is the space of endomorphisms of W commuting with the K-action.

We care about the case $k = \mathbb{R}$ and $K = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. Observe that a $\mathbb{C}$-vector space is simply an $\mathbb{R}$-vector space W with an endomorphism $J : W \to W$ such that $J^2 = -1 = -\operatorname{Id}_W$. A representation ρ satisfies

$$\rho(\phi)J = J\rho(\phi)$$

for all $\phi \in \mathfrak{Cl}(V, q)$. In more sophisticated language: A $\mathbb{C}$-representation of $\mathfrak{Cl}(V, q)$ is a vector space W that is a representation of $\mathfrak{Cl}(V, q)$ (as an $\mathbb{R}$-algebra) and a representation of $\mathbb{C}$ (as an $\mathbb{R}$-algebra) such that both representations commute.

Similarly, a $\mathbb{H}$-vector space W is an $\mathbb{R}$-vector space with endomorphisms I, J, K satisfying the usual relations for quaternions, and an $\mathbb{H}$-representation of $\mathfrak{Cl}(V, q)$ is one commuting with the action of $\mathbb{H}$ on W.

Any complex representation of $\mathfrak{Cl}_{r,s}$ automatically extends to a representation of $\mathfrak{Cl}_{r,s} \otimes \mathbb{C} \cong \mathfrak{Cl}_{r+s}$. (The isomorphism is because every $\mathfrak{Cl}_{r,s}$ is a direct sum of matrix algebras, and $\mathbb{R}(n) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}(n)$.) Similarly, any quaternionic representation of $\mathfrak{Cl}_{r,s}$ is automatically complex (with even complex dimension) because $\mathbb{C}$ is a subalgebra of $\mathbb{H}$.

Example 1. Recall that $\mathfrak{Cl}_2 \cong \mathbb{H}$. $\mathfrak{Cl}_2$ acts on itself by left multiplication, which gives a 4-dimensional irreducible representation W of $\mathfrak{Cl}_2$. We will show shortly that every representation of $\mathfrak{Cl}_2$ is isomorphic to an iterated direct sum of copies of W.

[1] The book says "field containing k," but $\mathbb{H}$ isn't a field, so I prefer this terminology.

By the previous remark, W can also be viewed as a 2-dimensional complex representation. More specifically, write $\mathbb{C} = \mathrm{span}_{\mathbb{R}}\{1, i\} \cong \mathrm{span}_{\mathbb{R}}\{1, e_1\}$. It follows that $\mathfrak{Cl}_2 \cong \mathrm{span}_{\mathbb{C}}\{1, e_2\}$. The action of e_1 is given by

$$e_1 \cdot 1 = e_1 = \begin{bmatrix} i \\ 0 \end{bmatrix}$$

$$e_1 \cdot e_2 = e_1 e_2 = -e_2 e_1 = \begin{bmatrix} 0 \\ i \end{bmatrix}$$

and of e_2 by

$$e_2 \cdot 1 = e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$e_2 \cdot e_2 = -1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

We can describe this representation in terms of complex matrices via

$$e_1 \mapsto \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad e_2 \mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad e_1 e_2 \mapsto \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$$

These matrices anticommute and square to -1, so they generate an algebra isomorphic to the quaternions.

2 Classification

Definition 2. Let $\rho : \mathfrak{Cl}(V, q) \to \hom_K(W, W)$ be a K-representation of the Clifford algebra $\mathfrak{Cl}(V, q)$. It is *reducible* if it splits as a direct sum

$$W = W_1 \oplus W_2$$

with the W_i invariant under the action of K and of $\mathfrak{Cl}(V, q)$. It is *irreducible* if no such representation exists.

These modules are usually called *decomposable*, but Clifford algebras are semisimple, so the two properties are equivalent. We will discuss why they are semisimple later on.

It follows by easy induction on dimension that all representations of a semisimple algebra can be written as direct sums of irreducibles. We are therefore interested in classifying all the irreps (irreducible representations) up to isomorphism (a linear map commputing with the representation.)

Lemma 1. *Let $K = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ and consider the $\mathbb{R}$-algebra $K(n)$ of $n \times n$ matrices. $K(n)$ has a natural representation on K^n, and up to equivalence this is the only irrep.*

Proof. Matrix algebras are *simple*: the have no nontrivial proper ideals. It's a fundamental result in representation theory that simple algebras have a single representation up to isomorphism. □

We can now use the classification of Clifford algebras (which we haven't done yet?) and this lemma to write down the irreducible representations. We mostly care about $\mathfrak{Cl}_n$ and $\mathfrak{Cl}_{n,\mathbb{C}}$. (**Warning:** the book uses notation for the complex Clifford algebra that's easy to misread, so I'm using a different one.) Here's a summarized version of Table III:

n	$\mathfrak{Cl}_n$	v_n	d_n	K	$\mathfrak{Cl}_{n,\mathbb{C}}$	$v_n^{\mathbb{C}}$	$d_n^{\mathbb{C}}$
1	$\mathbb{C}$	1	2	$\mathbb{C}$	$\mathbb{C} \oplus \mathbb{C}$	2	1
2	$\mathbb{H}$	1	4	$\mathbb{H}$	$\mathbb{C}(2)$	1	2
3	$\mathbb{H} \oplus \mathbb{H}$	2	4	$\mathbb{H}$	$\mathbb{C}(2) \oplus \mathbb{C}(2)$	2	2
4	$\mathbb{H}(2)$	1	8	$\mathbb{H}$	$\mathbb{C}(4)$	1	4
5	$\mathbb{C}(4)$	1	8	$\mathbb{C}$	$\mathbb{C}(4) \oplus \mathbb{C}(4)$	2	4
6	$\mathbb{R}(8)$	1	8	$\mathbb{R}$	$\mathbb{C}(8)$	1	8
7	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	2	8	$\mathbb{R}$	$\mathbb{C}(8) \oplus \mathbb{C}(8)$	2	8
8	$\mathbb{R}(16)$	1	16	$\mathbb{R}$	$\mathbb{C}(16)$	1	16

where K indicates the largest of $\mathbb{R}, \mathbb{C}, \mathbb{H}$ with which the representation commutes, v_n, $v_n^{\mathbb{C}}$ count the number of distinct irreps, and d_n, $d_n^{\mathbb{C}}$ give the dimension of the irreps over $\mathbb{R}$ and $\mathbb{C}$, respectively. (If there are two irreps it's the same for both.)

This table tells you everything you'd want to know, because of the periodicity isomorphisms

$$\begin{aligned}
\mathfrak{Cl}_{n+8,0} &\cong \mathfrak{Cl}_{n,0} \otimes \mathfrak{Cl}_{8,0} \\
\mathfrak{Cl}_{0,n+8} &\cong \mathfrak{Cl}_{0,n} \otimes \mathfrak{Cl}_{0,8} \\
\mathfrak{Cl}_{n+2,\mathbb{C}} &\cong \mathfrak{Cl}_{n,\mathbb{C}} \otimes \mathfrak{Cl}_{2,\mathbb{C}}
\end{aligned}$$

You can see how this pattern works in the complex case in the above table. The number of irreps doesn't change as you get to higher levels of the periodicity, but their dimensions increase.

3 Representations of Spin and Pin

3.1 Motivating example

Let's consider the historically important case of Spin_3. Recall that Spin_3 consists of the unit-norm elements of $\mathfrak{Cl}_3^0 \cong \mathfrak{Cl}_2 \cong \mathbb{H}$. There are many ways of describing the quaternions, and one is to represent $\mathbf{i}, \mathbf{j}, \mathbf{k}$ as the matrices

$$\sigma_1 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

$$\sigma_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\sigma_3 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

(If you divide these by i they're called the *Pauli matrices* in physics.) One reason to do this is that they satisfy

$$[\sigma_1, \sigma_2] = -2\sigma_3$$

and cyclic permutations of the indices. These are (up to some constants) the commutation relations for the Lie algebras $\mathfrak{su}_2 \cong \mathfrak{so}_3$. We know from earlier that Spin_3 is a double cover of SO(3), which means that they should have the same Lie algebra; we confirmed above that there is an isomorphism of Lie algebras $\mathfrak{Cl}_3^0 = \mathfrak{cl}_3^0 \cong \mathfrak{so}_3$.

Suppose you want to find all the representations of $\mathfrak{so}_3$, in order to find the representations of SO(3). $\mathfrak{so}_3$ can be defined as the Lie algebra of skew-symmetric 3×3 real matrices, and this algebra has an obvious representation $W = \mathbb{R}^3$ by multiplication. It turns out that the symmetric powers $\mathrm{Sym}^n W$ are all irreducible representations.

However, these are only half the representations. $\mathfrak{su}_2 \cong \mathfrak{so}_3$, and $\mathfrak{su}_2$ is an algebra of 2×2 matrices, so it has a two-dimensional representation V. It can be shown that the symmetric powers of V are also all irreducible. Furthermore $\mathrm{Sym}^2 V \cong W$, so these include the powers of W. In fact, every irreducible representation of $\mathfrak{su}_2$ is a symmetric power of V.

We wanted to find the representations of SO(3), but we've actually found more: the representation $\mathrm{Sym}^n V$ of $\mathfrak{so}_3$ lifts to a representation of SO(3) exactly when n is an even number. The remaining odd representations do, however, lift to Spin_3. Half of the representations lift because Spin_3 is a double cover of SO(3).

One perspective on Clifford algebras, spin groups, etc. is to make this construction systematic.

3.2 General case

Actually, what we *really* care about are representations of Spin.

Definition 3. The *real spinor representation* of Spin_n is the representation

$$\Delta_n : \mathrm{Spin}_n \to \mathrm{GL}(S) \tag{1}$$

given by restricting an irreducible real representation S of $\mathfrak{Cl}_n^0$.

To think about these, it's helpful to recall that

$$\mathfrak{Cl}_{r,s} \cong \mathfrak{Cl}_{r+1,s}^0$$

and consequently that

$$\mathfrak{Cl}_n(\mathbb{C}) \cong \mathfrak{Cl}_{n+1}^0(\mathbb{C}).$$

In particular, $\mathrm{Spin}_n \subset \mathfrak{Cl}_n^0 \cong \mathfrak{Cl}_{n-1}$, so the irreducible representations of $\mathfrak{Cl}_{n-1}$ are important in constructing the irreps of Spin_n. To construct them we'll want to use the volume form:

Definition 4. Consider $\mathfrak{Cl}_n$ with the underlying vector space spanned by $e_1, \ldots, e_n$. The *volume element* is defined by

$$\omega = e_1 \cdots e_n$$

and for $\mathfrak{Cl}_{n,\mathbb{C}}$ the *complex volume element* is

$$\omega_{\mathbb{C}} = i^{\frac{n+1}{2}} \omega.$$

These are well-defined up a sign, which can be fixed by a choice of ordered basis.

Proposition 2. *For $\mathfrak{Cl}_n$,*

1. $\omega = \omega_{\mathbb{C}}$ *exactly when* $n \equiv 7, 8 \pmod 8$
2. *If* n *is odd,* ω *and* $\omega_{\mathbb{C}}$ *are central*
3. $\omega^2 = 1$ *if* $n \equiv 3, 4 \pmod 4$
4. $(\omega_{\mathbb{C}})^2 = 1$ *for all* n

As a consequence, we have decompositions

$$\begin{aligned} \mathfrak{Cl}_n &= \mathfrak{Cl}_n^+ \oplus \mathfrak{Cl}_n^-, && n \equiv 3 \pmod 4 \\ \mathfrak{Cl}_{n,\mathbb{C}} &= \mathfrak{Cl}_{n,\mathbb{C}}^+ \oplus \mathfrak{Cl}_{n,\mathbb{C}}^-, && n \text{ odd} \end{aligned}$$

by using the projectors $1 \pm \omega$:

$$\mathfrak{Cl}_n^\pm = (1 \pm \omega)\mathfrak{Cl}_n$$

and similarly for the complex case. Notice that the nontrivial direct sums occur exactly in the dimensions where there are two irreps.

Example 2. Consider the case of $\mathrm{Spin}_2 \subset \mathfrak{Cl}_2^0 \cong \mathfrak{Cl}_1 \cong \mathbb{C}$. We can write Spin_2 as the set $\{a + be_1e_2 : a^2 + b^2 = 1\} \cong S^1$. $\mathfrak{Cl}_2$ as a whole is isomorphic to $\mathbb{H}$. We want to see how this representation restricts to $\mathfrak{Cl}_2^0$. Observe that

$$\begin{aligned} (e_1e_2) \cdot 1 &= e_1e_2 \\ (e_1e_2) \cdot e_1e_2 &= -1 \\ (e_1e_2) \cdot e_1 &= e_2 \\ (e_1e_2) \cdot e_2 &= -e_1 \end{aligned}$$

We see that the left $\mathfrak{Cl}_2$-module $\mathfrak{Cl}_2$ splits as a left $\mathfrak{Cl}_2^0$-module into $\mathfrak{Cl}_2^0$ and $\mathfrak{Cl}_2^1$, which are isomorphic modules of $\mathbb{R}$-dimension two. In both cases, these are simply the action of S^1 on $\mathbb{R}^2$ by rotations. The representation Δ_2 of Spin_2 is the direct sum $\mathbb{R}^2 \oplus \mathbb{R}^2$.

Proposition 3. *Let* W *be an irreducible real representation of* $\mathfrak{Cl}_n$ *for* $n = 4m + 3$. *Then* ω *acts on* W *by either* 1 *or* -1*, and the corresponding representations are nonisomorphic. Both possibilities occur.*

The analagous statements hold for $\mathfrak{Cl}_{n,\mathbb{C}}$ *and* n *odd.*

Proof. $\rho(\omega^2) = \rho(\omega)^2 = \mathrm{Id}$, so we can decompose W into (± 1)-eigenspaces. Since ω is central, these eigenspaces are $\mathfrak{Cl}_n$-invariant, so one of them must be all of W by irreducibility. (That is: apply Schur's lemma.) To construct the representations in the first place, we can find them as irreducible factors in the left action of $\mathfrak{Cl}_n$ on $\mathfrak{Cl}_n^+$ and $\mathfrak{Cl}_n^-$, respectively. □

Example 3. Now consider $\mathrm{Spin}_3 \subset \mathfrak{Cl}_3^0 \cong \mathfrak{Cl}_2 \cong \mathbb{H}$. We saw previously that it consists of the unit quaternions. As a whole, $\mathfrak{Cl}_3$ is isomorphic to $\mathbb{H} \oplus \mathbb{H}$, hence has two irreducible representations, one for each factor.

We didn't actually prove this decomposition, but the easiest way is to use the volume element $\omega = e_1 e_2 e_3$. $1+\omega = 1 + e_1e_2e_3$ has kernel

$$W^- = \mathrm{span}_{\mathbb{R}}\{1 - e_1e_2e_3, e_1 + e_2e_3, e_2 + e_1e_3, e_3 - e_1e_2\}$$

which is the same as saying that these are the -1-eigenvectors for ω. W^+ is the kernel of $1-\omega$, which is obtained by swapping the signs above. W^+ and W^- are the two distinct irreducible representations of $\mathfrak{Cl}_3$.

Recall that we already have a name for switching the signs: it's the action of the automorphism α of $\mathfrak{Cl}_3$, which is defined by $\alpha(e_i) = -e_i$. In particular, α sends $W^\pm$ to $W^\mp$, and we can write $\mathfrak{Cl}_3^0$ as the set of vectors of the form $\phi + \alpha(\phi)$ for $\phi \in \mathfrak{Cl}_3^+$. It follows that W^+ and W^- are isomorphic as representations of $\mathfrak{Cl}_3^0$. Restricting either representation to Spin_3 shows that Δ_3 is the usual representation of $\mathrm{Spin}_3 \cong S^3$ on $\mathbb{R}^4$ by unit quaternions. (We aren't focusing on Pin, but I'm pretty sure that as Pin_3 representations W^+ and W^- are distinct.)

Proposition 4. *Let W be an irreducible real representation of $\mathfrak{Cl}_n$ for $n = 4m$, and consider the splitting*

$$W = W^+ \oplus W^-$$

for $W^\pm = (1 \pm \omega)W$. Each of the subspaces $W^\pm$ is invariant under $\mathfrak{Cl}_n^0$, and under the isomorphism $\mathfrak{Cl}_n^0 \cong \mathfrak{Cl}_{n-1}$ these spaces correspond to the two distinct irreducible real representations.

The analagous statements hold for $\mathfrak{Cl}_{n,\mathbb{C}}$ and n even.

Proof. To see that $W^\pm$ are invariant under $\mathfrak{Cl}_n^0$, notice that ω commutes with everything in $\mathfrak{Cl}_n^0$: moving ω past an even product of generators contributes no overall sign. Under the isomorphism $\mathfrak{Cl}_{n-1} \cong \mathfrak{Cl}_n^0$, the volume element $\omega' = e_1 \cdots e_{n-1}$ of $\mathfrak{Cl}_{n-1}$ goes to the volume element ω of $\mathfrak{Cl}_n^0$, because

$$(e_1e_n)\cdots(e_{n-1}e_n) = (-1)^{\frac{1}{2}(n-1)(n-2)} e_1 \cdots e_{n-1} e_n^{n-1} = e_1 \cdots e_n$$

since

$$\frac{1}{2}(n-1)(n-2) + (n-2) = \frac{n(n-2)}{2}$$

is even when $n = 4m$. □

Example 4. $\mathfrak{Cl}_4$ is large enough that I don't want to work out the example by hand, but it has a single representation V of dimension 8 over $\mathbb{R}$ (given by the algebra of 2×2 invertible quaternionic matrices.) Because ω commutes with $\mathfrak{Cl}_4^0$, the ±1-eigenspaces $W^\pm$ of ω are both irreps when we restrict to $\mathfrak{Cl}_4^0$. One way to see that they're distinct is that $\omega \in \mathfrak{Cl}_4^0$. As a consequence, the representation Δ_4 of Spin_4 has two distinct irreducible factors of dimension 4.

Proposition 5. *When $n \equiv 3 \pmod 4$ the definition of Δ_n is independent of which irrep of $\mathfrak{Cl}_n$ is used. For $n \not\equiv 0 \pmod 4$ the representation Δ_n is either irreducible or a direct sum of two equivalent irreps; the second possibility occurs exactly when $n \equiv 1, 2 \pmod 8$. In the other cases, there is a decomposition*

$$\Delta_{4m} = \Delta_{4m}^+ \oplus \Delta_{4m}^-$$

into distinct irreducible representations of Spin_{4m}.

In summary:

n	Δ_n
1	two identical factors
2	two identical factors
3	one factor, doesn't depend on choice of irrep
4	two different factors
5	one factor
6	one factor
7	one factor, doesn't depend on choice of irrep
8	two different factors

Proof. We have previously discussed several cases. □

For completeness, we mention the complex case.

Definition 5. The *complex spin representation* of Spin_n is the homomorphism

$$\Delta_n^{\mathbb{C}} : \mathrm{Spin}_n \to \mathrm{GL}_{\mathbb{C}}(S)$$

given by restricting an irreducible complex representation of $\mathfrak{Cl}_{n,\mathbb{C}}$ to $\mathrm{Spin}_n \subset \mathfrak{Cl}_n^0 \subset \mathfrak{Cl}_{n,\mathbb{C}}$.

When n is odd, there are two irreps of $\mathfrak{Cl}_{n,\mathbb{C}}$ to restrict, but the representation $\Delta_n^{\mathbb{C}}$ doesn't depend on which one we pick and is irreducible. When n is even, $\Delta_{2m}^{\mathbb{C}} \cong \Delta_{2m}^{\mathbb{C}+} \oplus \Delta_{2m}^{\mathbb{C}-}$ is a direct sum of two distinct irreps.

3.3 Additional properties

There are some more facts that are worth mentioning, given time.

Proposition 6. *The Lie subalgebra of $\mathfrak{cl}_n$ (which is the vector space $\mathfrak{Cl}_n$ with the Lie bracket coming from the algebra multiplication) corresponding to the subgroup $\mathrm{Spin}_n \subset \mathfrak{Cl}_n^\times$ is*

$$\mathfrak{spin}_n = \wedge^2 \mathbb{R}^n$$

that is the image of the canonical embedding $\wedge^2\mathbb{R}^n \hookrightarrow \mathfrak{Cl}_n$.

Proof. For $i < j$, consider the curve

$$\begin{aligned}\gamma(y) &= (e_i \cos t + e_j \sin t)(-e_i \cos t + e_j \sin t)\\ &= (\cos^2 t - \sin^2 t) + 2e_ie_j \sin t \cos t\\ &= \cos(2t) + \sin(2t)e_ie_j\end{aligned}$$

It lies in Spin_n and has $\gamma'(0) = e_ie_j$, so $\mathfrak{spin}_n$ contains the image of $\wedge^2\mathbb{R}^n$. By dimension-counting we can conclude it's exactly the image. □

On reason to care about this: The Lie algebra of SO_n is the space of skew-symmetric transformations of $\mathbb{R}^n$. Such maps are naturally given by elements of $\wedge^2\mathbb{R}^n$, via

$$(v \wedge w)(x) = \langle v, x\rangle w - \langle w, x\rangle v$$

and we can therefore see directly that $\mathfrak{spin}_n \cong \mathfrak{so}_n$.

Proposition 7. *Let W be a real representation of $\mathfrak{Cl}_n$. Then there is an inner product $\langle\cdot,\cdot\rangle$ on W invariant under the action of the generators of $\mathfrak{Cl}_n$. If W can be extended to a complex or quaternionic representation, the product can be chosen to be invariant under that extension.*

Proof. Consider the *Clifford group* $F_n \subset \mathfrak{Cl}_n^\times$ generated by $e_1, \ldots, e_n$. It's a finite group, and the Clifford algebra is almost the group algebra $\mathbb{R}F_n$:

$$\mathfrak{Cl}_n \cong \mathbb{R}F_n/\langle -1+1\rangle$$

In particular, a representation of $\mathfrak{Cl}_n$ is exactly a representation of F_n in which -1 acts by $-\mathrm{Id}$. (This is another reason why Clifford algebras are semisimple, at least I think.)

Thus, given our representation W, we can choose a K-invariant inner product (where $K = \mathbb{R}, \mathbb{C}, \mathbb{H}$) and average over F_n to get an F_n-invariant, hence $\mathfrak{Cl}_n$-invariant inner product. $\square$

Proposition 8. *When $n \equiv 2, 6 \pmod 8$ Δ_n is a unitary representation, and when $n \equiv 3, 4, 5 \pmod 8$ it is symplectic.*

Proof. Δ_n comes from a representation of $\mathfrak{Cl}_n^0 \cong \mathfrak{Cl}_{n-1}$; when the representations of $\mathfrak{Cl}_{n-1}$ are complex we get unitary representations and when they are quaternionic we get symplectic ones. (It's not quite clear to me why this is: the book doesn't say more.) $\square$