X — n-dim manifold
E, F — smooth, complex vector bundles over X.

Def’n. A **differential operator of order m** on X is a linear map, \( P: \Gamma(E) \to \Gamma(F) \) with the given property: for each point in X there is a neighborhood U with local coordinates \((x_1, ..., x_n)\) and local trivializations of \(E|_U\) and \(F|_U\) where we can write,

\[
P = \sum_{|\alpha| = m} A^\alpha(x) \frac{\partial^{\alpha}}{\partial x^\alpha}
\]

multi-index notation

and where \( A^\alpha \neq 0 \) for some \( \alpha \) such that \(|\alpha| = m\).

Def’n. Given local coordinates & local trivializations of E and F, the **principal symbol** of the operator \( P \) (given above) is,

\[
\sigma(x, \xi)(P) = \sigma_\xi(P) : E_x \to F_x
\]

\[
\sigma_\xi(P) = i^m \sum_{|\alpha| = m} A^\alpha(x) \xi^\alpha
\]

**Question:** Is \( \sigma_\xi(P) \) well defined? Given different local coords and local trivializations is the symbol different?

There are two common solutions to the above,

1. Give a coord free definition of the principal symbol and show they're the same **not true for lower order.**
2. Show that \( \sum A^\alpha \) transform as a tensor under
Defn. a differential operator $P$ is elliptic if for each non-zero vector $\xi \in T^*_x X$ the principal symbol is invertible.

Example. Laplace-Beltrami operator, $\Delta: C^\infty(X) \rightarrow C^\infty(X)$

Let $X$ be endowed with some Riemannian metric $g$ with metric tensor $\Sigma g_{ij} dx_i dx_j$.

$$\Delta f = \frac{1}{\sqrt{g}} \Sigma \frac{2}{\partial x_j} (\frac{1}{\sqrt{g}} g^{ik} \frac{\partial f}{\partial x^k})$$

$$= \Sigma g^{ik} \frac{\partial^2 f}{\partial x_i \partial x_k} + \text{lower order terms}$$

$$\sigma_\xi(\Delta) = -\Sigma g^{ik} \xi_i \xi_k - ||\xi||^2$$

Example. Dirac operator, $D: \Gamma(S) \rightarrow \Gamma(S)$ over Riemannian $X$.

Recall, orthonormal basis of $\mathbb{R}^n$.

$$D\sigma = \Sigma e_j \cdot \nabla e_j \sigma$$

Choosing appropriate localizations,

$$D = \Sigma e_j \frac{\partial}{\partial x_j} + \text{constant terms}$$
\[ \sigma_\xi(D) = i \xi e_j \xi_j = i \xi \]

**Exercise.** Show that \( D^2: \Gamma(S) \to \Gamma(S) \) is elliptic.

**Prop.** Let \( P, P': \Gamma(E) \to \Gamma(F) \) (same order) and \( Q: \Gamma(F) \to \Gamma(L) \) be differential operators, then for \( \xi \in T^*_x X \) and \( t, t' \in \mathbb{R} \)
one has,

\[
\sigma_\xi(tP + t'P') = t \sigma_\xi(P) + t' \sigma_\xi(P') \\
\sigma_\xi(Q \circ P) = \sigma_\xi(Q) \circ \sigma_\xi(P)
\]

"The principal symbol is a NICE object when considered to live on the cotangent bundle."

Let's examine this fact in order to define an element of K-theory for an elliptic operator.

Recall (from K-theory) the equivalence of functors, \( \chi: \mathcal{L}(X,Y) \to K(X,Y) \) which equates the sequence

\[
\left[ V_0, ..., V_n, \sigma_1, ..., \sigma_n \right] \to O \to V_0 \to ... \to V_n \to O
\]

of vector bundles \( V_i \) over \( X \), with an element of K-theory.

This is how I'll denote the element of \( K(X,Y) \).

\[
\pi: T^*X \to X \\
P: \Gamma(E) \to \Gamma(F) \\
\sigma(P): \pi^*E \to \pi^*F \quad \text{(bundle map)}
\]

Prove: \( P \) is elliptic \( \Leftrightarrow \sigma(P) \) (above) is an isomorphism.
The symbol of an elliptic operator $P$ defines an element of $K$-theory,

\[ \sigma(P) = [\pi^*E, \pi^*F; \sigma(P)] \in K(DX, \partial DX) \cong K_{cpt}(TX) \]

\[ \text{by } DX/\partial DX \cong T^*X^+ \cong TX^+ \]

Now consider two maps:

1. the smooth embedding, $f: TX \hookrightarrow \mathbb{T}R^n$
2. the canonical "scrunch", $q: \mathbb{T}R^n \to \{\text{pt}\}$

Both of these maps induce maps on $K$-groups,

\[ f_!: K_{cpt}(TX) \to K_{cpt}(\mathbb{T}R^n) \]
\[ q_!: K_{cpt}(\mathbb{T}R^n) \to K(\text{pt}) \cong \mathbb{Z} \]

Def'n. the topological index of a differential operator $P$ is the integer,

\[ \text{top-Ind}(P) = q_!(f_!(\sigma(P))) \]