Differential operators and topological index

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X — n-dim manifold

E, F — smooth, complex vector bundles over X.

<u>Def'n</u>. A differential operator of order m on X is a linear map, $P : \Gamma(E) \to \Gamma(F)$ with the given property: for each point in X there is a nbhd U with local coordinates $(x_1, \dots, x_n)$ and local trivializations of $E|_U$ and $F|_U$ where we can write,

$$\boxed{P = \sum_{|\alpha| \le m} A^\alpha(x) \frac{\partial^{|\alpha|}}{\partial x^\alpha}}$$

multi-index notation

and where $A^\alpha \neq 0$ for some α such that $|\alpha| = m$.

<u>Def'n</u>. Given local coordinates & local trivializations of E and F, the principal symbol of the operator P (given above) is,

$$\sigma(x,\xi)(P) = \sigma_\xi(P) : E_x \to F_x$$

$$\boxed{\sigma_\xi(P) = i^m \sum_{|\alpha| = m} A^\alpha(x) \xi^\alpha}$$

<u>Question</u>: Is $\sigma_\xi(P)$ well defined? Given different local coords and local trivializations is the symbol different?

There's two common solutions to the above,

(1) Give a coord free definition of the principal symbol and show they're the same — not true for lower order.

(2) Show that $\{A^\alpha\}_{|\alpha| = m}$ transform as a tensor under

changes of coordinate system $\Rightarrow \sigma(P) \cdot \odot^m TX \otimes E \to F$
$\Rightarrow \sigma(P) \in \Gamma(\odot^m TX \otimes \mathrm{Hom}(E,F))$.

this is what the book does.

Def'n. a differential operator P is elliptic if for each non-zero vector $\xi \in T_x^* X$ the principal symbol is invertible.

sections of trivial $\mathbb{C}$-bundle of X.

Example. Laplace-Beltrami operator, $\Delta : C^\infty(X) \longrightarrow C^\infty(X)$

Let X be endowed with some Riemannian metric g with metric tensor $\sum_i g_{jk}\, dx_j\, dx_k$.

$$\Delta f = \frac{1}{\sqrt{g}} \sum_i \frac{\partial}{\partial x_j}\left(\sqrt{g}\, g^{jk} \frac{\partial f}{\partial x^k}\right)$$

$$= \sum_i g^{jk} \frac{\partial^2 f}{\partial x_j \partial x_k} + \text{lower order terms}$$

$$\sigma_\xi(\Delta) = -\sum_i g^{jk} \xi_j \xi_k = -\|\xi\|^2$$

Example. Dirac operator, $D : \Gamma(S) \longrightarrow \Gamma(S)$ Dirac bundle over Riemannian X.

Recall, orthonormal basis of $\mathbb{R}^n$.

$$D\sigma = \sum_i e_j \cdot \nabla_{e_j} \sigma$$

Choosing appropriate localizations,

$$D = \sum_i e_j \frac{\partial}{\partial x_j} + \text{constant terms}$$

$\sigma_\xi(D) = i\sum_j e_j \xi_j = i\xi$

<u>Exercise</u>. Show that $D^2 : \Gamma(S) \to \Gamma(S)$ is elliptic.

<u>Prop</u>. Let $P, P' : \Gamma(E) \to \Gamma(F)$ (same order) and $Q : \Gamma(F) \to \Gamma(L)$ be differential operators, then for $\xi \in T^*_x X$ and $t, t' \in \mathbb{R}$ one has,

$$\sigma_\xi(tP + t'P') = t\sigma_\xi(P) + t'\sigma_\xi(P')$$
$$\sigma_\xi(Q \circ P) = \sigma_\xi(Q) \circ \sigma_\xi(P)$$

"The principal symbol is a NICE object when considered to live on the cotangent bundle."

Let's examine this fact in order to define an element of K-theory for an elliptic operator.

Recall (from K-theory) the equivalence of functors, $\chi : L(X,Y) \to K(X,Y)$ which equates the sequence

$$[V_0, \dots, V_n ; \sigma_1, \dots, \sigma_n] = 0 \to V_0 \to \cdots \to V_n \to 0$$

(*) exact when restricted to Y.

of vector bundles V_i over X, with an element of K-theory.

this is how I'll denote the element of $K(X,Y)$.

$\pi : T^*X \to X$

$P : \Gamma(E) \to \Gamma(F)$

$\sigma(P) : \pi^*E \to \pi^*F$ (bundle map)

Prop. P is elliptic $\Leftrightarrow$ $\sigma(P)$ (above) is an isomorphism away

Prop. P is elliptic $\Leftrightarrow$ $\sigma(P)$ (above) is an isomorphism away from the zero section.

$DX = \{\xi \in T^*X \mid \|\xi\| \leq 1\}$

The symbol of an elliptic operator P defines an element of K-theory,

$$\underline{\sigma}(P) = [\pi^*E, \pi^*F; \sigma(P)] \in K(DX, \partial DX) \simeq K_{cpt}(TX)$$

$\hookrightarrow DX/\partial DX \underset{\text{homeo.}}{\simeq} T^*X^+ \simeq TX^+$

Now consider two maps:

(1) the smooth embedding, $f: TX \hookrightarrow T\mathbb{R}^N$

(2) the canonical "scrunch", $q: T\mathbb{R}^N \to \{pt\}$

Both of these maps induce maps on K-groups,

$$f_!: K_{cpt}(TX) \to K_{cpt}(T\mathbb{R}^N)$$
$$q_!: K_{cpt}(T\mathbb{R}^N) \to K(pt) \simeq \mathbb{Z}.$$

Def'n. the topological index of a differential operator P is the integer,

$$\boxed{\text{top-ind}(P) = q_!(f_!(\underline{\sigma}(P)))}$$