Talk 1: Clifford alg introduction

Friday, November 30, 2018 5:03 PM

Clifford Algebras

Def. Let V be a vector space over a field k with quadratic form q. Let
$$\mathcal{T}(V) = \bigoplus_{n\geq 0} V^{\otimes n}$$
be the tensor algebra of V and let $\mathcal{I}_q(V)$ be the ideal generated by $v\otimes v + q(v)1$ for all $v \in V$. Then the Clifford algebra is the quotient,
$$Cl(V,q) = \mathcal{T}(V)/\mathcal{I}_q(V)$$

The algebra $Cl(V,q)$ is generated by the vector space $V \subset Cl(V,q)$ (and 1), and subject to the relations,
$$v\cdot v = -q(v)\cdot 1.$$
for $v \in V$. If the characteristic of k is not 2, then for all $v, w \in V$
$$v\cdot w + w\cdot v = -2q(v,w)$$

ex. Compute $Cl(\mathbb{R})$ (with the usual inner product).

just 1 as the 1-dim vector space.

We have two generators, 1 and e_1, and the relation $e_1^2 = -1$.
So $Cl(\mathbb{R}) \cong \mathbb{C}$ by the mapping,
$$1 \longmapsto 1$$

$e_1 \mapsto i$

ex. Compute $Cl(\mathbb{R}^2)$.

There are four generators: $1, e_1, e_2, e_1e_2$;
and the relations are $e_1^2=-1$, $e_2^2=-1$ and
$(e_1e_2)^2=-1$

use $e_1e_2+e_2e_1=0$

$$e_1e_2=-e_2e_1$$
$$e_1e_2e_1=e_2$$
$$(e_1e_2)^2=-1.$$

and so $Cl(\mathbb{R}^2)\simeq\mathbb{H}$ (the quaternions) by the map,

$$1\mapsto 1$$
$$e_1\mapsto i$$
$$e_2\mapsto j$$
$$e_1e_2\mapsto k$$

Prop (the Universal Property) Let $f: V\to A$ be a linear map to an associative k-algebra with unit, such that

$$f(v)\cdot f(v) = -q(v)\cdot 1$$

for all $v\in V$. Then f extends uniquely to a k-algebra homomorphism $\tilde{f}: Cl(V,q)\to A$. Furthermore, $Cl(V,q)$ is the unique associative k-algebra with this property.

$\boxed{\mathbb{Z}_2\text{-grading on } Cl(V,q)}$

Consider the automorphism of $Cl(V,q)$, α, which extends the map $\alpha(v) = -v$ on V.
There is a decomposition, even part

$$Cl(V,q) = Cl_0(V,q) \oplus Cl_1(V,q)$$ odd part.

where $Cl_i(V,q) = \{\varphi \in Cl(V,q) : \alpha(\varphi) = (-1)^i \varphi\}$ are the eigenspaces of α. In this way we can regard $Cl(V,q)$ as a $\mathbb{Z}_2$-graded algebra.

<u>Prop</u>. there is a canonical vector space isomorphism,

$$\Lambda^* V \xrightarrow{\sim} Cl(V,q)$$

compatible with the canonical filtrations.

caution: NOT an algebra isomorphism.

<u>Prop</u>. Let $V = V_1 \oplus V_2$ be a q-orthogonal decomposition. Then there is a natural isomorphism of Clifford algebras

$$Cl(V,q) \longrightarrow Cl(V_1,q_1) \hat{\otimes} Cl(V_2,q_2)$$

where q_i denotes the restriction of q to V_i and where $\hat{\otimes}$ denotes the $\mathbb{Z}_2$-graded tensor product.

The Transpose

The tensor algebra, $\mathcal{T}(V)$, has an involution given by, $v_1 \otimes \cdots \otimes v_r \mapsto v_r \otimes \cdots \otimes v_1$. This map preserves the ideal and descends to a map,

$$(\ )^t : Cl(V,q) \to Cl(V,q)$$

called the transpose. This is an antiautomorphism, i.e. $(\varphi\psi)^t = \psi^t \varphi^t$.

The Algebras Cl_n and $Cl_{r,s}$

We define the algebra $Cl_{r,s} \equiv Cl(V,q)$ where $V = \mathbb{R}^{r+s}$ and

$$q(x) = x_1^2 + \dots + x_r^2 - x_{r+1}^2 - \dots - x_{r+s}^2$$

We denote the special cases, $Cl_n \equiv Cl_{n,0}$ and $Cl_n^* \equiv Cl_{0,n}$.

Prop. Let $e_1, \dots, e_{r+s}$ be an orthonormal basis of $\mathbb{R}^{r+s}$. Then $Cl_{r,s}$ is generated (as an algebra) by $e_1, \dots, e_{r+s}$ subject to the relations,

$$e_ie_j + e_je_i = \begin{cases} -2\delta_{ij} & \text{if } i \leq r \\ 2\delta_{ij} & \text{if } i > r. \end{cases}$$

Prop. There is an isomorphism,

$$Cl_{r,s} \cong \underbrace{Cl_1 \hat{\otimes} \cdots \hat{\otimes} Cl_1}_{r \text{ times}} \hat{\otimes} \underbrace{Cl_1^* \hat{\otimes} \cdots \hat{\otimes} Cl_1^*}_{s \text{ times}}$$

Which follows inductively from the previously mentioned proposition.

We already computed $Cl_1 = Cl(\mathbb{R})$ and $Cl_2 = Cl(\mathbb{R}^2)$. Let's compute a few more examples.

ex. Compute $Cl_1^* = Cl_{0,1}$.

Two generators: $1, e_1$ and the relation $e_1^2 = 1$.

Thus it is clear that $Cl_1^* \cong \mathbb{R} \oplus \mathbb{R}$.

ex. Compute $Cl_2^* = Cl_{0,2}$.

We have the generators: $1, e_1, e_2, e_1e_2$ and the relations, $e_1^2 = 1$, $e_2^2 = 1$ and $e_1e_2 = -e_2e_1$

$$(e_1e_2)^2 = -1$$

Consider the mapping:

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$e_1 \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$e_2 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$e_1e_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

So we see that $Cl_2^* \cong \mathbb{R}(2)$.

ex. Compute $Cl_{1,1}$.

Generators: $1, e_1, e_2, e_1e_2$ and relations: $e_1^2 = -1$, $e_2^2 = 1$, and $e_1e_2 = -e_2e_1 \Rightarrow (e_1e_2)^2 = 1$.

Following the example above it is clear that $Cl_{1,1} \cong \mathbb{R}(2)$.

Thm. There is an algebra isomorphism $Cl_{r,s} \cong Cl^0_{r+1,s}$ for all r and s.

Proof.

Choose an orthogonal basis $\{e_1, \dots, e_{r+s+1}\}$ of $\mathbb{R}^{r+s+1}$ such that $q(e_i) = 1$ for $1 \leq i \leq r+1$ and $q(e_i) = -1$ for $r+1 < i \leq r+s+1$.

Let $\mathbb{R}^{r+s} = \text{span}\{e_i \mid i \neq r+1\}$ and define a map, $f: \mathbb{R}^{r+s} \to Cl^0_{r+1,s}$ by setting $f(e_i) = e_{r+1}e_i$ for

$i \neq r+1$ (and extend linearly).

For $x = \sum_{i \neq r+1} x_i e_i$ we have,

$$f(x)^2 = \sum_{i,j} x_i x_j e_{r+1} e_i e_{r+1} e_j$$
$$= \sum_{i,j} x_i x_j e_i e_j = x \cdot x = -q(x) \cdot 1.$$

It follows from the universal property that f extends to an algebra homomorphism,

$$\tilde{f} : Cl_{r+s} \to Cl^0_{r+1,s}$$

Checking $\tilde{f}$ on a linear basis shows that $\tilde{f}$ is in fact an isomorphism.

□

Thm. There are isomorphisms,

$$Cl_{n,0} \otimes Cl_{0,2} \cong Cl_{0,n+2}$$
$$Cl_{0,n} \otimes Cl_{2,0} \cong Cl_{n+2,0}$$
$$Cl_{r,s} \otimes Cl_{1,1} \cong Cl_{r+1,s+1}$$

for all $n, r, s \geq 1$.

Proof.

Let's prove only the first case above, as the others follow in a similar manner.

Let $\{e_1, \ldots, e_{n+2}\}$ be an orthonormal basis of $\mathbb{R}^{n+2}$ with inner product $q(x) = -\|x\|^2$. Let $e_1', \ldots, e_n'$ be the standard basis of $Cl_{n,0}$ and e_1'', e_2'' be the standard basis of $Cl_{0,2}$. Define a map $f : \mathbb{R}^{n+2} \to Cl_{n,0} \otimes Cl_{0,2}$ by

$$f(e_i) = \begin{cases} e_i' \otimes e_1'' e_2'' & \text{for } 1 \leq i \leq n \\ 1 \otimes e_{i-n}'' & \text{for } i = n+1, n+2 \end{cases}$$

(and extend linearly). It can be shown that $f(x)^2 = \|x\|^2 \cdot 1 \otimes 1$, so by the universal property f extends to an algebra homomorphism

$\sim$

$f: Cl_{0,n+2} \to Cl_{n,0} \otimes Cl_{0,2}$.

Since $\tilde{f}$ maps onto a set of generators it is surjective and since $\dim Cl_{0,n+2} = \dim Cl_{n,0} \otimes Cl_{0,2}$, the map must be an isomorphism.

$\square$

<u>Thm</u>. For all $n \geq 0$, there are periodicity isomorphisms,

$$Cl_{n+8,0} \cong Cl_{n,0} \otimes Cl_{8,0}$$

$$Cl_{0,n+8} \cong Cl_{0,n} \otimes Cl_{0,8}$$

$$\mathbb{C}l_{n+2} \cong \mathbb{C}l_n \otimes_{\mathbb{C}} \mathbb{C}l_2$$

$\mathbb{C}l_n = Cl(\mathbb{C}^n, q_{\mathbb{C}})$

$= Cl_n \otimes_{\mathbb{C}} \mathbb{C}$.