Principal G-bundles, Clifford bundles, and spin or bundles

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<u>Def</u>. A principal G-bundle is a fiber bundle $\pi : P \to X$, together with a continuous right action of G on P which preserves the fibers by acting freely and transitively on them.

Locally P looks like $U \times G$ where G acts by multiplication on the right,

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\sim"] \arrow[dr] & G & U \times G \arrow[dl] \\ & U & \end{tikzcd}$$

<u>Ex</u>. Let X be any topological space, and let P be a two-sheeted cover of X, the P is a principal $\mathbb{Z}_2$-bundle, where the action of $\mathbb{Z}_2$ on P interchanges the sheets.

First we will give an example of a principal G-bundle that comes from a vector bundle and then give a general construction of a principal G-bundle coming from a fiber bundle.

<u>Ex</u>. Let $\begin{matrix} E \\ \downarrow \\ X \end{matrix}$ be a real, n-dim vector bundle, then one can define $P_{GL}(E)$, the bundle of bases in E. The fiber $P_{GL}(E)_x$ consist of the set of all bases of E_x. This is a principal GL_n-bundle.

The group action is as follows: fix $g = ((a_{ij})) \in GL_n$, then given a basis $p = (v_1, \dots, v_n) \in P_{GL}(E_x)$ (it's inherently in the fiber) we define $p \cdot g = (v_1', \dots, v_n')$ where $v_j' = \sum_k v_k a_{kj}$. It's easy to see that this action is free and transitive on the fibers.

If E is orientable we can construct a principal GL_n^+-bundle, $P_{GL^+}(E)$.

Associated principal G-bundle construction.

Consider a fiber bundle
$$\begin{tikzcd} B \arrow[d] & F \arrow[l] \\ X & \end{tikzcd}$$
with structure group $G \subseteq \mathrm{Homeo}(F)$.

Let's recall the definition,

i. $\{U_\alpha\}$ open cover of X

ii. $h_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\sim} U_\alpha \times F$ (local trivialization)

iii. $h_\alpha \circ h_\beta^{-1} : (U_\alpha \cap U_\beta) \times F \to (U_\alpha \cap U_\beta) \times F$

$$h_\alpha \circ h_\beta^{-1}(x, f) = (x, \underbrace{g_{\alpha\beta}(x)}f)$$ this is the LEFT action of $G \curvearrowright F$.

$g_{\alpha\beta}(x) : U_\alpha \cap U_\beta \to G$ (transition function)

Transition data satisfies, $g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1$.

Def. any fiber bundle with structure group G has an associated principal G-bundle $P_G(B)$ obtained in the following manner,

i. replace F by G in the local product

ii. paste $\{U_\alpha \times G\}$ together by the same transition functions, where $g_{\alpha\beta}(x) \in G$ acts on G on the LEFT. be ac

Note: while each fiber "looks like" G, there is no preferred identity.

Clifford and spinor bundles.

Ex. Let $\begin{matrix} E \\ \downarrow \\ X \end{matrix}$ be a real n-dim vector bundle over manifold X. Assume the bundle is oriented and equipped with a Riemannian structure. Then we can consider the bundle of oriented orthonormal frames, This is a principal SO_n-bundle.

Let $\pi : P \to X$ be a principal G-bundle and consider another space F. homomorphism, $\rho : G \to Homeo(F)$ we can construct a Fiber bundle over F as follows.

Consider the action of G on $P \times F$ given by $\varphi_g(p,f) = (p \cdot g^{-1}, \rho(g) \cdot f)$.

Def. Let $P \times_\rho F$ be the quotient space of the above action. One s projection $P \times F \to P \to X$ descends to a map $\pi_\rho : P \times_\rho F \to X$, which is th over X with fiber F called the bundle associated to P by ρ.

Some thoughts to consider ...

$SO_n \circlearrowright \mathbb{R}^n \xrightarrow[\text{Universal property}]{} SO_n \circlearrowright Cl_n \xrightarrow[\text{preserves multiplication}]{} cl(\rho_n) : SO_n \to Aut(Cl_n)$

Def. the Clifford bundle of the oriented Riemannian vector bundle E is the bundle

$$\boxed{Cl(E) = P_{SO}(E) \times_{cl(\rho_n)} Cl_n}$$

We could also define the $Cl(E)$ as the quotient bundle,

$$Cl(E) = \left(\Sigma \otimes^r E\right) / \mathcal{I}(E)$$

$\perp(E)$

<u>Fact</u>. $Cl(E)$ is a bundle of Clifford algebras over X.

<u>Fact</u>. each of the notions intrinsic to Clifford algebras carries over to Clifford bundles, i.e. the decomposition $Cl(E) = Cl^0(E) \oplus Cl^1(E)$

<u>Prop</u>. there is a canonical vector bundle isometry, $\lambda: \Lambda^*(E) \to Cl(E)$, under which $\lambda(\Lambda^{even}E) = Cl^0(E)$ and $\lambda(\Lambda^{odd}E) = Cl^1(E)$.

<u>Def</u>. Let E be an oriented Riemannian vector bundle with a spin structure $\xi: P_{spin}(E) \to P_{SO}(E)$. A real spinor bundle of E is a bundle of the form $S(E) = P_{spin}(E) \times_\mu M$, where M is a left module for Cl_n and where $\mu: Spin_n \to SO(M)$ is the representation given by left multiplication by elements of $Spin_n$.

<u>Ex</u>. Consider Cl_n as a module over itself by left multiplication ℓ.

The corresponding real spinor bundle is,

$$Cl_{spin}(E) = P_{spin}(E) \times_\ell Cl_n$$

Facts: $Cl_{spin}(E)$ is a principal Cl_n-bundle and there is a natural embedding $P_{spin}(E) \subset Cl_{spin}(E)$ hence every real spinor bundle for E can be captured from this one.

Note that $Cl_{spin}(E) \neq Cl(E)$. Let's see why, consider

$$Ad: Spin_n \to Aut(Cl_n)$$

$Ad_g(\varphi) = g\varphi g^{-1}$ $\quad$ $Ad_{-1} = \text{identity}$

$$Ad': SO_n \to Aut(Cl_n)$$

$= cl(\rho_n)$.

And so we see that $Cl(E) = P_{spin}(E) \times_{Ad} Cl_n$.

A worthy point. we care about reps μ that do NOT descend to reps of SO_n.

Prop. Let $S(E)$ be a real spinor bundle. Then $S(E)$ is a bundle of modules over the bundle of algebras $Cl(E)$. In particular the sections of $S(E)$ are a module over the sections of $Cl(E)$.

Proof.

The diagram,

$$\begin{tikzcd} P_{spin}(E) \times Cl_n \times M \arrow[r, "\mu"] \arrow[d, "\rho_g"] & P_{spin}(E) \times M \arrow[d, "\rho_g'"] \\ P_{spin}(E) \times Cl_n \times M \arrow[r, "\mu"] & P_{spin}(E) \times M \end{tikzcd}$$

given by,

$$\begin{tikzcd} (p, \varphi, m) \arrow[r, mapsto] \arrow[d, mapsto] \arrow[dr, phantom, "\circlearrowright"] & (p, \varphi \cdot m) \arrow[d, mapsto] \\ (p \cdot g^{-1}, g \cdot \varphi \cdot g^{-1}, g \cdot m) \arrow[r, mapsto] & (p \cdot g^{-1}, g \cdot \varphi \cdot m) \end{tikzcd}$$

Clearly commutes. Therefore μ descends to a mapping, $\mu : Cl(E) \oplus S(E) \longrightarrow S(E)$. Given by the map top left $\rightarrow$ bottom right of the above.

$\square$

Def. Two spinor bundles are equivalent iff they are equivalent as bundles of $Cl(E)$-modules. A bundle of $Cl(E)$-modules is called irreducible if at each x the fiber is irreducible as a module over $Cl(E_x)$.

<u>Prop.</u> Every spinor bundle of E can be decomposed into a direct sum of irreducible ones.