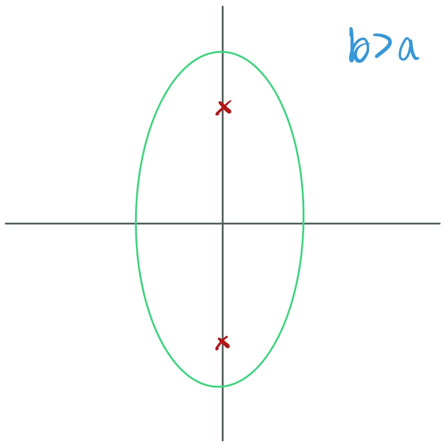
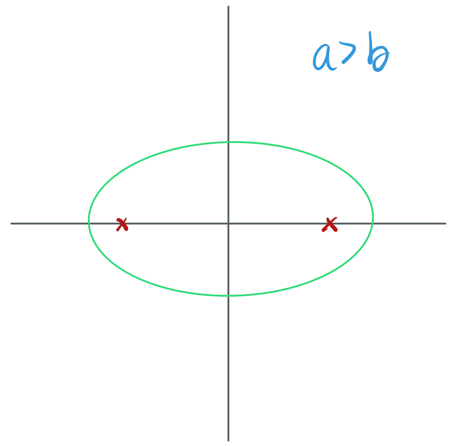
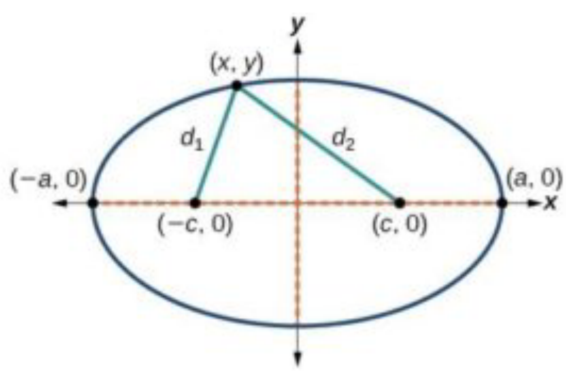
Recall the equation of an ellipse centered at $(0,0)$,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$





The sum of the distance from the foci to any point is constant.



<u>Fun fact you don't need to know.</u> $d_1+d_2=2a$ if $a>b$ and $d_1+d_2=2b$ if $b>a$.

Location of foci.

- $a > b$, $(\pm\sqrt{a^2-b^2}, 0)$
- $b > a$, $(0, \pm\sqrt{b^2-a^2})$

2.4 Polynomials.

A **polynomial** is a function p such that,

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n$$

where n is a non-negative integer.

We like to index the coefficients in this way because the power of x matches the corresponding coefficient index.
Note that $a_0 x^0 = a_0 1 = a_0$.

Let $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ where $a_n \neq 0$. The **degree of p**, denoted $\deg p$, is n.

Ex.

horizontal line$^{(*)}$ $\Rightarrow$ polynomial of degree 0.
line w/non-zero slope $\Rightarrow$ poly of degree 1.
quadratic $(a \neq 0)$ $\Rightarrow$ poly of degree 2.

(*) The horizontal line $p(x)=0$ is an exception.

Ex. $p(x) = x^3 - 2x + 1 \quad q(x) = x^3 + x^2 + 3x$

(A) What is $\deg p$? $\deg p = 3$

(B) What is $\deg(p+q)$?

$$(p+q)(x) = x^3 - 2x + 1 + x^3 + x^2 + 3x$$
$$= 2x^3 + x^2 + x + 1$$
$$\deg(p+q) = 3$$

(C) What is $\deg(p-q)$?

$$(p-q)(x) = x^3 - 2x + 1 - x^3 - x^2 - 3x$$
$$= -x^2 - 5x + 1$$
$$\deg(p-q) = 2$$

(D) What is $\deg(pq)$?

$$(pq)(x) = (x^3 - 2x + 1)(x^3 + x^2 + 3x)$$
$$= x^6 + x^5 + 3x^4 - 2x^4 - 2x^3 - 6x^2 + x^3 + x^2 + 3x$$
$$= x^6 + x^5 + x^4 - x^3 - 5x^2 + 3x$$
$$\deg(pq) = 6$$

The above should inspire some generalizations about degree...

Let p & q be NON-ZERO polynomials, then

(1) $\deg(pq) = \deg p + \deg q$

(2) $\deg(p \pm q) \leq \text{maximum}\{\deg p, \deg q\}$

Like with quadratics we are interested in the **zeros** (or roots) of a polynomial. That is, the values of x s.t. $p(x) = 0$.

Ex. Show that $p(x) = (x+1)(x^2+4)$ has a zero at $x = -1$.

Let's compute $p(-1)$,

$$p(-1) = (-1+1)((-1)^2+4)$$
$$= (0)(5) = 0.$$

We say that the polynomial above is **factored** because it is expressed as the product of polynomials with smaller degree.

A polynomial is **fully factored** if there is no further way we can decompose terms.

Some texts will call being fully factored just factored — context is important.

factors of p

$p(x) = (x+1)(x^2+4)$ is fully factored (over $\mathbb{R}$).

$x = t$ is a zero of $p(x)$ $\iff$ $(x-t)$ is a **factor** of t.

<u>Ex</u>. Find all zeros of $p(x) = (x-2)(x+3)(x^2-16)$

$(x-2)$ & $(x+3)$ are factors of $p(x)$ $\Rightarrow$ $x=2$ & $x=-3$ are zeros of $p(x)$.

All other zeros of $p(x)$ must be zeros of the last factor, x^2-16. So let's compute zeros of this,

$0 = x^2 - 16$
$16 = x^2$
$x = \pm 4$

How many zeros does a polynomial have?

- $0 \le \#\text{zeros} \le \deg p$
- if $\deg p$ is <u>odd</u>, p has at least one zero

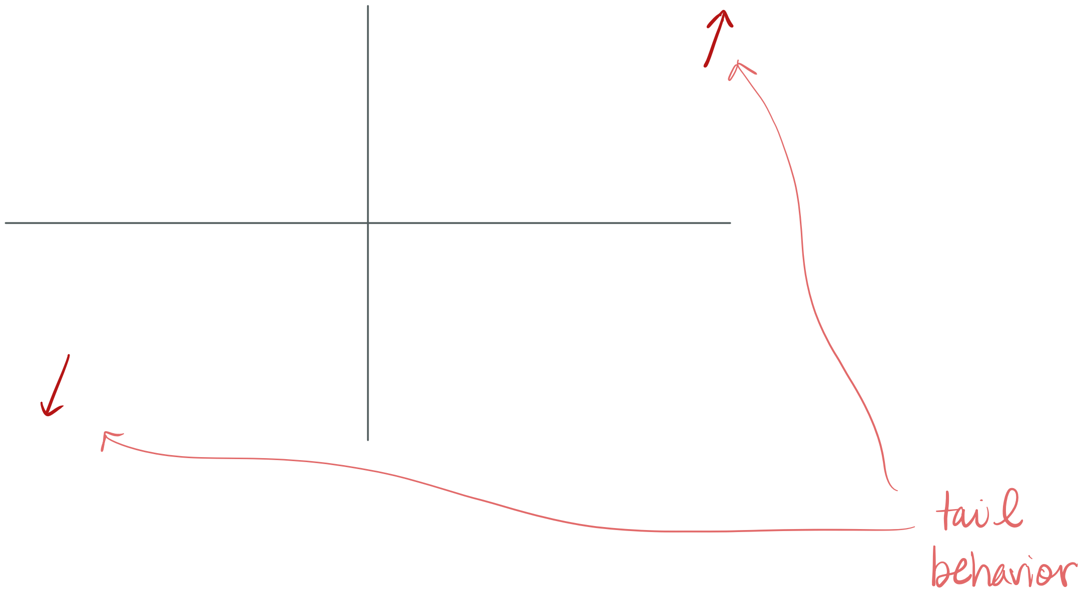
let's explore why this is true...

<u>Behavior of a polynomial as $x \to \pm\infty$</u>

Let $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ $(a_n \neq 0)$. Then p behaves like $a_n x^n$ as $|x|$ becomes large.

<u>Ex.</u> $p(x) = x^3 + x$

As $x \to \infty$, $p(x) \sim x^3 \to \infty$,
as $x \to -\infty$, $p(x) \sim x^3 \to -\infty$



Since the function is continuous, it must cross the x-axis